QFT 1 : Problem Set 1

1.) Peskin & Schroeder 2.1

We begin with the action for the classical electromagnetic field :

$$S\,[A] \;=\; \int d^4x\,\left(-\,\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu}\right) \qquad \text{where} \qquad F_{\mu\nu} \;=\; \partial_\mu A_\nu \,-\, \partial_\nu A_\mu$$

(a)

homogeneous Maxwell equations

To derive the homogeneous Maxwell equations we form the Poincare dual of the Faraday tensor.

$$G^{\alpha\beta} \;=\; \tfrac{1}{2}\,\epsilon^{\alpha\beta\mu\nu}\,F_{\mu\nu} \qquad F^{\alpha\beta} \;=\; -\,\tfrac{1}{2}\,\epsilon^{\alpha\beta\mu\nu}\,G_{\mu\nu}$$

G is divergenceless from the definition of F :

$$\partial_\alpha G^{\alpha\beta} \;=\; \epsilon^{\alpha\beta\mu\nu}\,\partial_\alpha\partial_\mu A_\nu \;=\; 0$$

From the expressions for the fields $(\mathbf{E},\mathbf{B})$ in terms of the potentials $A^\alpha \;=\; (\Phi,\mathbf{A})^\alpha$:

$$\mathbf{E} \;=\; -\,\frac{\partial\mathbf{A}}{\partial t} \,-\, \boldsymbol{\nabla}\Phi \qquad \mathbf{B} \;=\; \boldsymbol{\nabla}\times\mathbf{A}$$

We find (using Latin letters for spatial indices and Greek for spacetime indices) :

$$E^i \;=\; -\,\partial^0A^i \,+\, \partial^iA^0 \;=\; F^{i0}$$

$$B^i \;=\; -\,\epsilon^{0ijk}\partial_jA_k \;=\; -\tfrac{1}{2}\,\epsilon^{0ijk}\,F_{jk} \;=\; G^{i0}$$

Thus,

$$\partial_\alpha G^{\alpha 0} \;=\; \partial_i G^{i0} \;=\; \partial_i B^i \;=\; 0$$

$$\partial_\alpha G^{\alpha i} \;=\; \partial_0 G^{0i} \,+\, \partial_j\epsilon^{ji0k}\,F_{0k} \;=\; -\,\partial_0 B^i \,-\, \epsilon^{0ijk}\,\partial_j E^k \;=\; 0$$

Or,

$$\boldsymbol{\nabla}\cdot\mathbf{B} \;=\; 0 \qquad \frac{\partial\mathbf{B}}{\partial t} \,+\, \boldsymbol{\nabla}\times\mathbf{E} \;=\; 0$$

inhomogeneous Maxwell equations

We begin with the Euler-Lagrange equation :

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)}\right) \,-\, \frac{\partial\mathcal{L}}{\partial A_\nu} \;=\; 0$$

Now,

$$\frac{\partial\mathcal{L}}{\partial A_\nu} \;=\; 0 \qquad \text{and} \qquad \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} \;=\; -\,\tfrac{1}{2}\left(\delta^\mu{}_\alpha\delta^\nu{}_\beta \,-\, \delta^\mu{}_\beta\delta^\nu{}_\alpha\right)F^{\alpha\beta} \;=\; -\,F^{\mu\nu}$$

Thus,

$$\partial_\alpha F^{\alpha 0} \;=\; \partial_i F^{i0} \;=\; \partial_i E^i \;=\; 0$$

$$\partial_\alpha F^{\alpha i} \;=\; \partial_0 F^{0i} \,-\, \partial_j\epsilon^{ji0k}\,G_{0k} \;=\; -\,\partial_0 E^i \,+\, \epsilon^{0ijk}\,\partial_j B^k \;=\; 0$$

Or,

$$\boldsymbol{\nabla}\cdot\mathbf{E} \;=\; 0 \qquad \frac{\partial\mathbf{E}}{\partial t} \,-\, \boldsymbol{\nabla}\times\mathbf{B} \;=\; 0$$

(b)

We derive the Noether current associated with an infinitesimal translation $x \to x+a$. Using the equations of motion :

$$\delta\mathcal{L} \,=\, \frac{\partial\mathcal{L}}{\partial A_\nu}\,\delta A_\nu \,+\, \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)}\,\partial_\mu \delta A_\nu \,=\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)}\,\delta A_\nu\right)$$

From $\delta\mathcal{L} \,=\, a^\alpha \partial_\alpha \mathcal{L}$ and $\delta A_\nu \,=\, a^\alpha \partial_\alpha A_\nu$ we find $\partial_\alpha T^\alpha{}_\beta \,=\, 0$ where :

$$T^\alpha{}_\beta \,=\, \frac{\partial\mathcal{L}}{\partial\left(\partial_\alpha A_\nu\right)}\,\partial_\beta A_\nu \,-\, \delta^\alpha{}_\beta\,\mathcal{L}$$

Or,

$$T^\alpha{}_\beta \,=\, -\,F^{\alpha\nu}\partial_\beta A_\nu \,+\, \tfrac{1}{4}\,\delta^\alpha{}_\beta\,F_{\mu\nu}F^{\mu\nu}$$

Note that this tensor is not symmetric. It is also neither gauge invariant nor traceless. To remedy these problems we construct :

$$\widehat{T}^\alpha{}_\beta \,=\, T^\alpha{}_\beta \,+\, \partial_\lambda\left(F^{\alpha\lambda}A_\beta\right)$$

Using the equations of motion :

$$\widehat{T}_{\alpha\beta} \,=\, -\,F_\alpha{}^\nu\partial_\beta A_\nu \,+\, \tfrac{1}{4}\,\eta_{\alpha\beta}\,F_{\mu\nu}F^{\mu\nu} \,+\, F_\alpha{}^\nu\partial_\nu A_\beta$$

Or,

$$\widehat{T}_{\alpha\beta} \,=\, F_{\alpha\mu}F_{\nu\beta}\,\eta^{\mu\nu} \,+\, \tfrac{1}{4}\,\eta_{\alpha\beta}\,F_{\mu\nu}F^{\mu\nu}$$

We compute the energy and momentum densities in terms of the fields. Now,

$$F_{\mu\nu}F^{\mu\nu} \,=\, 2\,F_{0j}F^{0j} \,+\, F_{ij}F^{ij}$$

From above :

$$F^{0j} \,=\, -\,F_{0j} \,=\, -\,E^j \qquad\text{and}\qquad F^{ij} \,=\, F_{ij} \,=\, -\,\epsilon^{ij0k}\,G_{0k} \,=\, -\,\epsilon^{0ijk}\,B^k$$

Using $\epsilon^{0ijk}\,\epsilon^{0ijq} \,=\, 2\,\delta^{kq}$, we have :

$$F_{\mu\nu}F^{\mu\nu} \,=\, -\,2\,\left(\mathbf{E}^2 - \mathbf{B}^2\right)$$

We consider,

$$\mathcal{E} \,=\, \widehat{T}^{00} \,=\, F^{0\mu}F^{\nu 0}\,\eta_{\mu\nu} \,+\, \tfrac{1}{4}\,\eta^{00}\,F_{\mu\nu}F^{\mu\nu}$$

Or,

$$\mathcal{E} \,=\, \tfrac{1}{2}\,\left(\mathbf{E}^2 + \mathbf{B}^2\right)$$

Also,

$$S^i \,=\, \widehat{T}^{0i} \,=\, F^{0j}F^{ki}\,\eta_{jk} \,=\, E^j\,\epsilon^{ji0k}\,G_{0k} \,=\, \epsilon^{0ijk}\,E^j\,B^k$$

Or,

$$\mathbf{S} \,=\, \mathbf{E}\times\mathbf{B}$$

alternate derivation

We may also derive the symmetric, traceless and gauge invariant energy-momentum tensor from the action on a curved spacetime. The action is :

$$S\,[A,g] \;=\; \int d^4x\,\sqrt{g}\;\left(-\,\tfrac{1}{4}\,F_{\mu\nu}F_{\alpha\beta}\,g^{\mu\alpha}g^{\nu\beta}\,\right) \qquad \text{where} \qquad \sqrt{g} \;=\; \sqrt{|\det\,[\,g_{\mu\nu}\,]\,|}$$

Here the matrix of the components of a tensor in a particular coordinate system is represented by braces. For example :

$$[\,g_{\mu\nu}\,]^{-1} \;=\; [\,g^{\mu\nu}\,]$$

To compute the energy-momentum tensor we vary the action with respect to the metric with the definition :

$$T_{\alpha\beta}(x) \;\equiv\; \frac{2}{\sqrt{g}}\,\frac{\delta S\,[A,g]}{\delta g^{\alpha\beta}(x)}$$

Where,

$$S\,[A,g+\delta g]\;-\;S\,[A,g] \;\simeq\; \int d^4x\,\frac{\delta S\,[A,g]}{\delta g^{\alpha\beta}(x)}\,\delta g^{\alpha\beta}(x)$$

Now,

$$\det\,[\,g_{\mu\nu}\,] \;=\; \exp\left[\mathrm{tr}\,(\ln\,[\,g_{\mu\nu}\,])\right]$$

Thus,

$$\begin{aligned}
\det\,[\,g_{\mu\nu}+\delta g_{\mu\nu}\,] \;&=\; \det\,[\,g_{\mu\nu}\,]\;\det\Big(\mathbf{1}+[\,g_{\mu\nu}\,]^{-1}\,[\,\delta g_{\mu\nu}\,]\Big)\\
&=\; \det\,[\,g_{\mu\nu}\,]\exp\Big[\mathrm{tr}\,\Big(\ln\,\Big(\mathbf{1}+[\,g_{\mu\nu}\,]^{-1}\,[\,\delta g_{\mu\nu}\,]\Big)\Big)\Big]\\
&\simeq\; \det\,[\,g_{\mu\nu}\,]\,\Big(1+\mathrm{tr}\,\Big([\,g_{\mu\nu}\,]^{-1}\,[\,\delta g_{\mu\nu}\,]\Big)\Big)
\end{aligned}$$

Since,

$$\mathrm{tr}\,\Big([\,g_{\mu\nu}\,]^{-1}\,[\,\delta g_{\mu\nu}\,]\Big) \;=\; g^{\mu\nu}\,\delta g_{\mu\nu} \;=\; -\,g_{\mu\nu}\,\delta g^{\mu\nu}$$

We find :

$$\sqrt{g+\delta g} \;\simeq\; \sqrt{g}\;\left(1-\tfrac{1}{2}\,g_{\mu\nu}\,\delta g^{\mu\nu}\,\right)$$

This leads to:

$$S\,[A,g+\delta g]\;-\;S\,[A,g] \;\simeq\; -\,\frac{1}{2}\int d^4x\,\sqrt{g}\;\left(F_{\mu\nu}F_{\alpha\beta}\,g^{\mu\alpha}\;-\;\tfrac{1}{4}\,g_{\nu\beta}\,F_{\sigma\rho}F^{\sigma\rho}\right)\delta g^{\nu\beta}$$

Thus we have :

$$T_{\alpha\beta} \;=\; F_{\alpha\mu}F_{\nu\beta}\,g^{\mu\nu}\;+\;\tfrac{1}{4}\,g_{\alpha\beta}\,F_{\mu\nu}F^{\mu\nu}$$

This clearly reduces to the above result (derived via Noethers' theorem) when restricted to flat spacetime.

2.) Peskin & Schroeder 2.2 : The complex scalar field

We begin with the action for the complex scalar field :

$$S[\phi] = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right)$$

(a)

We compute the Hamiltonian density associated with this action:

$$\mathcal{H} = \pi_\phi \dot{\phi} + \pi_{\phi^*} \dot{\phi}^* - \mathcal{L}$$

Where,

$$\pi \equiv \pi_\phi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^* \qquad \text{and} \qquad \pi_{\phi^*} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi} = \pi^*$$

Thus since,

$$\mathcal{L} = \dot{\phi}^* \dot{\phi} - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi$$

We find :

$$\mathcal{H} = \pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi$$

We now impose the canonical commutation relations :

$$[\, \phi(\mathbf{x}, t)\, ,\, \pi(\mathbf{y}, t)\,] = i\, \delta^3(\mathbf{x} - \mathbf{y})$$

All other commutators (except the one given by hermitian conjugation) vanish. We use these and the Heisenberg equations of motion to verify that $\dot{\phi} = \pi^*$:

$$\begin{aligned} i\, \dot{\phi}(x) &= [\, \phi(x)\, ,\, H\,] = \int d^3y\, [\, \phi(\mathbf{x}, t)\, ,\, \mathcal{H}(\mathbf{y}, t)\,] \\ &= i\, \dot{\phi}(x) = \int d^3y\, \pi^*(\mathbf{y}, t)\, [\, \phi(\mathbf{x}, t)\, ,\, \pi(\mathbf{y}, t)\,] = i\, \pi^*(x) \end{aligned}$$

We now consider $\dot{\pi} = \ddot{\phi}^*$:

$$\begin{aligned} i\, \dot{\pi}(x) &= [\, \pi(x)\, ,\, H\,] = \int d^3y\, [\, \pi(\mathbf{x}, t)\, ,\, \mathcal{H}(\mathbf{y}, t)\,] \\ &= \int d^3y\, \left[\, \pi(\mathbf{x}, t)\, ,\, \left(\nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \right) (\mathbf{y}, t)\, \right] \\ &= -i \int d^3y\, \left(\nabla \phi^*(\mathbf{y}, t) \cdot \nabla_{\mathbf{y}} + m^2 \phi^*(\mathbf{y}, t) \right) \delta^3(\mathbf{y} - \mathbf{x}) \end{aligned}$$

Integrating by parts we find :

$$\dot{\pi}(x) = \nabla \cdot \nabla \phi^*(x) - m^2 \phi^*$$

Thus ϕ satisfies the Klein-Gordon equation :

$$\ddot{\phi} = \dot{\pi}^* = \nabla^2 \phi - m^2 \phi \qquad \text{or} \qquad \partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

(b)

By analogy with the real scalar field we postulate the following form for $\phi(x)$:

$$\phi(x) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\,E_{\mathbf{p}}}}\,\left(\,a_{\mathbf{p}}\,e^{-ip\cdot x} \,+\, b^\dagger_{\mathbf{p}}\,e^{ip\cdot x}\,\right)$$

Thus,

$$\pi(x) \;=\; -i\int \frac{d^3p}{(2\pi)^3}\,\sqrt{\frac{E_{\mathbf{p}}}{2}}\,\left(\,b_{\mathbf{p}}\,e^{-ip\cdot x} \,-\, a^\dagger_{\mathbf{p}}\,e^{ip\cdot x}\,\right)$$

Again by analogy with the case of the real Klein-Gordon field we postulate :

$$\left[\,a_{\mathbf{p}}\,,\,a^\dagger_{\mathbf{k}}\right] \;=\; (2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{k}) \qquad \text{and} \qquad \left[\,b_{\mathbf{p}}\,,\,b^\dagger_{\mathbf{k}}\right] \;=\; (2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{k})$$

We assume all other commutators vanish and verify these relations by computing :

$$\begin{aligned}
[\,\phi(\mathbf{x},t)\,,\,\pi(\mathbf{y},t)\,] \;&=\; \frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{k}}}{E_{\mathbf{p}}}}\,\left[\,a_{\mathbf{p}}\,,\,a^\dagger_{\mathbf{k}}\right]\,e^{-ip\cdot x}\,e^{ik\cdot y} \\
&+\; \frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{p}}}{E_{\mathbf{k}}}}\,\left[\,b_{\mathbf{p}}\,,\,b^\dagger_{\mathbf{k}}\right]\,e^{ik\cdot x}\,e^{-ip\cdot y} \\
&=\; i\,\delta^3(\mathbf{x}-\mathbf{y})
\end{aligned}$$

We now show that H is diagonal when written in terms of these creation and annihilation operators. We begin with :

$$H \;=\; \int d^3x\,\left(\pi^\dagger\pi \,+\, \nabla\phi^\dagger\cdot\nabla\phi \,+\, m^2\phi^\dagger\phi\right)$$

We consider each of the terms that make up H in turn. Since H is a Noether charge, we may evaluate it at $t=0$) without loss of generality :

$$\begin{aligned}
&\int d^3x\;\pi^\dagger(\mathbf{x},0)\pi(\mathbf{x},0) \\
&\quad=\; \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\sqrt{E_{\mathbf{p}}E_{\mathbf{k}}}\int d^3x\,\left(b^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} \,-\, a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(b_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} \,-\, a^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \\
&\quad=\; \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\left(b^\dagger_{\mathbf{p}}b_{\mathbf{p}} \,+\, a_{\mathbf{p}}a^\dagger_{\mathbf{p}} \,-\, b^\dagger_{\mathbf{p}}a^\dagger_{-\mathbf{p}} \,-\, a_{\mathbf{p}}b_{-\mathbf{p}}\right)
\end{aligned}$$

Now,

$$\nabla\phi \;=\; i\int\frac{d^3p}{(2\pi)^3}\,\frac{\mathbf{p}}{\sqrt{2\,E_{\mathbf{p}}}}\,\left(a_{\mathbf{p}}\,e^{i\mathbf{p}\cdot\mathbf{x}} \,-\, b^\dagger_{\mathbf{p}}\,e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$

Thus,

$$\begin{aligned}
&\int d^3x\;\nabla\phi^\dagger(\mathbf{x},0)\,\cdot\,\nabla\phi(\mathbf{x},0) \\
&\quad=\; \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\,\frac{\mathbf{p}\cdot\mathbf{k}}{\sqrt{E_{\mathbf{p}}E_{\mathbf{k}}}}\int d^3x\,\left(a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} \,-\, b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} \,-\, b^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \\
&\quad=\; \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\,\frac{\mathbf{p}^2}{E_{\mathbf{p}}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} \,+\, b_{\mathbf{p}}b^\dagger_{\mathbf{p}} \,+\, a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}} \,+\, b_{\mathbf{p}}a_{-\mathbf{p}}\right)
\end{aligned}$$

Also,

$$
\begin{aligned}
&\int d^3x\; m^2\, \phi^\dagger(\mathbf{x},0)\,\phi(\mathbf{x},0) \\
&\quad = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\,\frac{m^2}{\sqrt{E_\mathbf{p}E_\mathbf{k}}}\int d^3x\,\left(a^\dagger_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}} + b_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(a_\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}} + b^\dagger_\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \\
&\quad = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\frac{m^2}{E_\mathbf{p}}\left(a^\dagger_\mathbf{p}a_\mathbf{p} + b_\mathbf{p}b^\dagger_\mathbf{p} + a^\dagger_\mathbf{p}b^\dagger_{-\mathbf{p}} + b_\mathbf{p}a_{-\mathbf{p}}\right)
\end{aligned}
$$

Combining terms, taking $\mathbf{p}\to-\mathbf{p}$ for cross terms and using $E^2_\mathbf{p} = \mathbf{p}^2 + m^2$:

$$
\begin{aligned}
&\int d^3x\,\left(\nabla\phi^\dagger\cdot\nabla\phi + m^2\phi^\dagger\phi\right)(\mathbf{x},0) \\
&\qquad = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,E_\mathbf{p}\left(a^\dagger_\mathbf{p}a_\mathbf{p} + b_\mathbf{p}b^\dagger_\mathbf{p} - b^\dagger_\mathbf{p}a^\dagger_{-\mathbf{p}} - a_\mathbf{p}b_{-\mathbf{p}}\right)
\end{aligned}
$$

Thus,

$$
H = \int d^3x\,\mathcal{H}(\mathbf{x},0) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,E_\mathbf{p}\left(a^\dagger_\mathbf{p}a_\mathbf{p} + a_\mathbf{p}a^\dagger_\mathbf{p} + b^\dagger_\mathbf{p}b_\mathbf{p} + b_\mathbf{p}b^\dagger_\mathbf{p}\right)
$$

Finally, we normal order to remove the infinite energy of the so-called Dirac sea.

$$
:H: = \int \frac{d^3p}{(2\pi)^3}\,E_\mathbf{p}\left(a^\dagger_\mathbf{p}a_\mathbf{p} + b^\dagger_\mathbf{p}b_\mathbf{p}\right)
$$

(c)

We now express the $U(1)$ Noether charge in terms of creation and annihilation operators :

$$
Q = \frac{i}{2}\int d^3x\,\left(\phi^\dagger\pi^\dagger - \pi\phi\right)
$$

Again the charge is conserved so we may evaluate it at $t=0$:

$$
\begin{aligned}
&\int d^3x\;\pi(\mathbf{x},0)\,\phi(\mathbf{x},0) \\
&\quad = \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{E_\mathbf{p}}{E_\mathbf{k}}}\int d^3x\,\left(a^\dagger_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}} - b_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(a_\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}} + b^\dagger_\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \\
&\quad = \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\left(a^\dagger_\mathbf{p}a_\mathbf{p} - b_\mathbf{p}b^\dagger_\mathbf{p} + a^\dagger_\mathbf{p}b^\dagger_{-\mathbf{p}} - b_\mathbf{p}a_{-\mathbf{p}}\right)
\end{aligned}
$$

Since $(\pi\phi)^\dagger = \phi^\dagger\pi^\dagger$:

$$
\begin{aligned}
&\int d^3x\;\phi^\dagger(\mathbf{x},0)\,\pi^\dagger(\mathbf{x},0) \\
&\qquad = \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\left(b_\mathbf{p}b^\dagger_\mathbf{p} - a^\dagger_\mathbf{p}a_\mathbf{p} + a^\dagger_\mathbf{p}b^\dagger_{-\mathbf{p}} - b_\mathbf{p}a_{-\mathbf{p}}\right)
\end{aligned}
$$

Thus,

$$
Q = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left(a^\dagger_\mathbf{p}a_\mathbf{p} - b_\mathbf{p}b^\dagger_\mathbf{p}\right)
$$

Upon normal ordering, we see that the (a,b) particles have charge $\left(+\frac{1}{2},-\frac{1}{2}\right)$:

$$
:Q: = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left(a^\dagger_\mathbf{p}a_\mathbf{p} - b^\dagger_\mathbf{p}b_\mathbf{p}\right)
$$

(d)

We consider the Lagrangian :

$$\mathcal{L} \,=\, \partial_\mu \phi_a^* \, \partial^\mu \phi_a \,-\, m^2 \phi_a^* \phi_a$$

We will first consider the general case $a = 1 \ldots N$ and then take $N = 2$. We rewrite the Lagrangian in terms of an N dimensional complex vector ϕ and its hermitian conjugate $\phi^\dagger$:

$$\mathcal{L} \,=\, \partial_\mu \phi^\dagger \, \partial^\mu \phi \,-\, m^2 \phi^\dagger \phi$$

This is invariant under a global $U(N)$ transformation :

$$\phi \,\rightarrow\, U\phi \qquad\qquad \text{where} \qquad\qquad U^\dagger U \,=\, \mathbf{1}$$

We may decompose any $U(N)$ transformation into a $U(1)$ and an $SU(N)$ transformation. Given $U \in U(N)$ such that $\det U = e^{i\varphi}$ we may form $M = e^{-i\varphi/N}\, U \in SU(N)$. Thus we may consider these invariances separately. For $U(1)$ we consider $\phi \rightarrow \tilde{\phi}(\alpha) = e^{-i\alpha/2}\phi$. Since the Lagrangian itself, rather than merely the action, is invariant under the $U(1)$, we have the conserved current :

$$J^\mu \;=\; \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \left.\frac{\partial \tilde{\phi}}{\partial \alpha}\right|_{\alpha=0} \;+\; \left.\frac{\partial \tilde{\phi}^\dagger}{\partial \alpha}\right|_{\alpha=0} \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^\dagger\right)}$$

Now,

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \,=\, \partial^\mu \phi^\dagger \qquad\qquad \text{and} \qquad\qquad \left.\frac{\partial \tilde{\phi}}{\partial \alpha}\right|_{\alpha=0} \,=\, -\tfrac{i}{2}\,\phi$$

Thus,

$$J^\mu \;=\; -\tfrac{i}{2}\left(\partial^\mu \phi^\dagger \phi \,-\, \phi^\dagger \partial^\mu \phi\right)$$

And,

$$Q \;\equiv\; \int d^3x\, J^0 \;=\; \frac{i}{2} \int d^3x \left(\phi^\dagger \pi^\dagger \,-\, \pi\phi\right)$$

Here we have :

$$\pi \;\equiv\; \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \;=\; \dot{\phi}^\dagger$$

For $SU(N)$ we express each element of the group (here we work in the vector representation and its complex conjugate) in terms of the exponentiation of elements of the Lie algebra $su(N)$. That is if $M \in SU(N)$ then it can be expressed as $M = e^{-i\alpha^j g^j}$ where$g^j \in su(N)$ and $\alpha^j \in \mathbb{R}$. Thus since $U^\dagger U = 1$ we have $\left(g^j\right)^\dagger = g^j$ and since,

$$\ln\left(\det M\right) \,=\, \ln\left(\det\left(e^{-i\alpha^j g^j}\right)\right) \,=\, \mathrm{tr}\left(-i\alpha^j g^j\right) \,=\, 0$$

we see that g^j is traceless. Thus we are looking for a set of linearly independent traceless hermitian matrices in N dimensions. The dimensionality of this space is $\left(N^2 - 1\right)$.
Note: The true underlying invariance of the lagrangian is $O(2N)$ not $U(N)$. There are thus actually $N(2N-1)$, not N^2, symmetry generators.
For SU(N) we have the commutation relations :

$$\left[\, g^j \,,\, g^k \,\right] \,=\, i\, f^{jkl}\, g^l$$

We use the conventional normalizations :

$$\mathrm{tr}\left(g^j g^k\right) \,=\, \tfrac{1}{2}\,\delta^{jk} \qquad\qquad \text{and} \qquad\qquad f^{jmn}\, f^{kmn} \,=\, N\,\delta^{jk}$$

We consider the symmetry $\phi \to \tilde{\phi}(\boldsymbol{\alpha}) = e^{-i\alpha^j g^j}\phi$. This leads to the conserved currents :

$$\left(J^k\right)^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \left.\frac{\partial \tilde{\phi}}{\partial \alpha^k}\right|_{\alpha^k=0} + \left.\frac{\partial \tilde{\phi}^\dagger}{\partial \alpha^k}\right|_{\alpha^k=0} \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^\dagger\right)}$$

Now,

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} = \partial^\mu \phi^\dagger \qquad \text{and} \qquad \left.\frac{\partial \tilde{\phi}}{\partial \alpha^k}\right|_{\alpha^k=0} = -i\, g^k\, \phi$$

Thus,

$$\left(J^k\right)^\mu = -i\left(\partial^\mu \phi^\dagger g^k \phi - \phi^\dagger g^k \partial^\mu \phi\right)$$

And,

$$Q^k \equiv \int d^3x\, \left(J^k\right)^0 = i\int d^3x\left(\phi^\dagger\, g^k\, \pi^\dagger - \pi\, g^k\, \phi\right)$$

We now show that the charges satisfy the same commutation relations in their action on the Hilbert space as the generators of the Lie algebra satisfy on $\mathbb{C}^N$. As above, since the charges are conserved, we may evaluate the fields that go into their construction at any time. We will therefore suppress time labels on the fields in what follows. We first rewrite the charges with explicit $\mathbb{C}^N$ indices.

$$Q^j = i\int d^3x\left(\phi_a^\dagger(\mathbf{x})\, \left(g^j\right)_{ab}\, \pi_b^\dagger(\mathbf{x}) - \pi_a(\mathbf{x})\, \left(g^j\right)_{ab}\, \phi_b(\mathbf{x})\right)$$

We now evaluate the commutator :

$$\begin{aligned}\left[\, Q^j\, ,\, Q^k\,\right] = -\int d^3x \int d^3y \;\; &\Big(\left[\,\phi_a^\dagger(\mathbf{x})\, \left(g^j\right)_{ab}\, \pi_b^\dagger(\mathbf{x})\, ,\, \phi_c^\dagger(\mathbf{y})\, \left(g^k\right)_{cd}\, \pi_d^\dagger(\mathbf{y})\right] \\ &+ \left[\,\pi_a(\mathbf{x})\, \left(g^j\right)_{ab}\, \phi_b(\mathbf{x})\, ,\, \pi_c(\mathbf{y})\, \left(g^k\right)_{cd}\, \phi_d(\mathbf{y})\right]\Big)\end{aligned}$$

Now,

$$\left(\left[\,\phi^\dagger(\mathbf{x})\, \left(g^j\right)\, \pi^\dagger(\mathbf{x})\, ,\, \phi^\dagger(\mathbf{y})\, \left(g^k\right)\, \pi^\dagger(\mathbf{y})\,\right]\right) = -\left(\left[\,\pi(\mathbf{x})\, \left(g^j\right)\, \phi(\mathbf{x})\, ,\, \pi(\mathbf{y})\, \left(g^k\right)\, \phi(\mathbf{y})\,\right]\right)^\dagger$$

Thus we only need to evaluate :

$$\begin{aligned}&\left[\,\pi_a(\mathbf{x})\, \left(g^j\right)_{ab}\, \phi_b(\mathbf{x})\, ,\, \pi_c(\mathbf{y})\, \left(g^k\right)_{cd}\, \phi_d(\mathbf{y})\,\right] \\ &\qquad = \left(g^j\right)_{ab}\, \left(g^k\right)_{cd}\, \left(\,\pi_a(\mathbf{x})\left[\,\phi_b(\mathbf{x})\, ,\, \pi_c(\mathbf{y})\,\right]\, \phi_d(\mathbf{y}) + \pi_c(\mathbf{y})\, \left[\,\pi_a(\mathbf{x})\, ,\, \phi_d(\mathbf{y})\,\right]\, \phi_b(\mathbf{x})\,\right)\end{aligned}$$

Using,

$$\left[\,\phi_a(\mathbf{x})\, ,\, \pi_b(\mathbf{y})\,\right] = i\, \delta_{ab}\, \delta^3(\mathbf{x}-\mathbf{y})$$

We find :

$$\left[\,\pi_a(\mathbf{x})\, \left(g^j\right)_{ab}\, \phi_b(\mathbf{x})\, ,\, \pi_c(\mathbf{y})\, \left(g^k\right)_{cd}\, \phi_d(\mathbf{y})\,\right] = i\, \delta^3(\mathbf{x}-\mathbf{y})\, \left(\pi(\mathbf{x})\left[\, g^j\, ,\, g^k\,\right]\phi(\mathbf{y})\,\right)$$

Thus,

$$\left[\, Q^j\, ,\, Q^k\,\right] = -\,\epsilon^{jkl}\int d^3x\, \left(\phi^\dagger(\mathbf{x})\, g^l\, \pi^\dagger(\mathbf{x}) - \pi(\mathbf{x})\, g^l\, \phi(\mathbf{x})\,\right) = i\, f^{jkl}\, Q^l$$

For the case of $SU(2)$ we make the replacements :

$$g^j = \frac{\sigma^j}{2} \qquad \text{and} \qquad f^{jkl} = \epsilon^{jkl}$$

Where σ^j are the Pauli matrices and ϵ^{jkl} is the completely anti-symmetric tensor in 3 dimensions $\left(\epsilon^{123} = 1\right)$.

3.) Peskin & Schroeder 2.3

We evaluate the function

$$\langle 0|\,\phi(x)\phi(y)\,|0\rangle \;=\; D(x-y) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,e^{-ip\cdot(x-y)}$$

for spacelike $(x-y)$, such that $(x-y)^2 \;=\; -r^2$, explicitly in terms of Bessel functions. Since $D(x)$ is invariant under Lorentz transformations, $D(x) = D(\Lambda x)$ $(\Lambda \in SO\,(3,1))$, we may choose $x^0 \;=\; y^0$. Thus, denoting $\mathbf{x}-\mathbf{y}=\mathbf{r}$,

$$D(x-y) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,e^{i\mathbf{p}\cdot\mathbf{r}} \;=\; \frac{1}{(2\pi)^2}\int_0^{\pi} d\theta\,\sin\theta \int_0^{\infty} dp\,\frac{p^2}{2\,(p^2+m^2)^{1/2}}\,e^{ipr\cos\theta}$$

Where we have introduced a sperical coordinate system (p,θ,ϕ) such that $\mathbf{p}\cdot\mathbf{r} \;=\; pr\cos\theta$. Thus,

$$D(x-y) \;=\; \frac{1}{(2\pi)^2}\int_{-1}^{1} dx \int_0^{\infty} dp\,\frac{p^2}{2\,(p^2+m^2)^{1/2}}\,e^{iprx} \;=\; \frac{1}{(2\pi)^2}\,\frac{1}{r}\int_0^{\infty} dp\,\frac{p\,\sin(pr)}{(p^2+m^2)^{1/2}}$$

From,

$$K_1(x) \;=\; -\,\pi/2\;(J_1(ix)+iN_1(ix)) \;=\; \int_0^{\infty} du\,\frac{u}{(1+u^2)^{1/2}}\,\sin(xu)$$

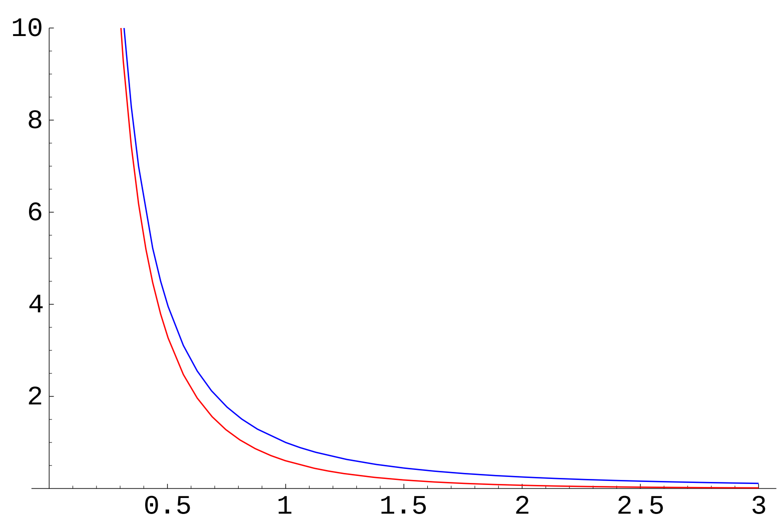
Defining $u \;=\; p/m$,

$$D(x-y) \;=\; \frac{1}{(2\pi)^2}\,\frac{m}{r}\int_0^{\infty} du\,\frac{u}{(1+u^2)^{1/2}}\,\sin(mru) \;=\; \frac{1}{(2\pi)^2}\,\frac{m}{r}\,K_1(mr)$$

From the properties of $K_1(x)$, we find that for (x,y) spacelike separated :

$$D(x-y) \;\sim\; \frac{1}{(2\pi)^2\,r^2} \qquad \text{as} \qquad x\to y$$

The following is a plot of $K_1(x)/x$ in red and $1/x^2$ in blue :



4.)

We consider coherent states for the real Klein-Gordon field :

$$|f\rangle \;=\; N_f \,\exp\left[\, i \int \frac{d^3p}{(2\pi)^3}\, f(\mathbf{p})\, a^\dagger_{\mathbf{p}} \right] |0\rangle$$

Where,

$$N_f \;=\; \exp\left[-\tfrac{1}{2} \int \frac{d^3p}{(2\pi)^3}\; |f(\mathbf{p})|^2 \right]$$

(a)

We evaluate the expectation value of the field operator :

$$\langle f|\, \phi(x)\, |f\rangle$$

Where,

$$\phi(x) \;=\; \phi_+(x) \,+\, \phi_-(x) \;=\; \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\,E_{\mathbf{p}}}} \left(a_{\mathbf{p}}\, e^{-ip\cdot x} \,+\, a^\dagger_{\mathbf{p}}\, e^{ip\cdot x} \right)$$

Since, $\phi_-(x) = \phi_+^\dagger(x)$ we need only evaluate :

$$\langle f|\, \phi_+(x)\, |f\rangle \;=\; |N_f|^2 \;\langle 0|\; e^{-i\Gamma^\dagger}\; \phi_+(x)\, e^{i\Gamma}\; |0\rangle$$

Where,

$$\Gamma \;=\; \int \frac{d^3p}{(2\pi)^3}\, f(\mathbf{p})\, a^\dagger_{\mathbf{p}}$$

Thus,

$$\langle f|\, \phi_+(x)\, |f\rangle \;=\; |N_f|^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\,E_{\mathbf{p}}}}\, e^{-ip\cdot x}\; \langle 0|\; e^{-i\Gamma^\dagger}\; a_{\mathbf{p}}\, e^{i\Gamma}\; |0\rangle$$

We now show that $|f\rangle$ is normalised so that :

$$\langle f|\; f\rangle \;=\; |N_f|^2 \;\langle 0|\; e^{-i\Gamma^\dagger}\; e^{i\Gamma}\; |0\rangle \;=\; 1$$

Now, if $[A, B] \in \mathbb{C}$ then:

$$e^A\, e^B \;=\; e^{A+B+\frac{1}{2}[A,B]} \;=\; e^B\, e^A e^{[A,B]}$$

Also,

$$\Gamma^\dagger\, |0\rangle \;=\; 0 \qquad \text{and} \qquad \left[\Gamma^\dagger, \Gamma\right] \;=\; \int \frac{d^3p}{(2\pi)^3}\; |f(\mathbf{p})|^2$$

Thus,

$$\langle f|\; f\rangle \;=\; |N_f|^2\; \exp\left[\int \frac{d^3p}{(2\pi)^3}\; |f(\mathbf{p})|^2 \right] \langle 0|\; e^{i\Gamma}\, e^{-i\Gamma^\dagger}\; |0\rangle \;=\; 1$$

Now,

$$[\, a_{\mathbf{p}}\, ,\, \Gamma\,] \;=\; f(\mathbf{p}) \qquad \text{since} \qquad \left[\, a_{\mathbf{p}}\, ,\, a^\dagger_{\mathbf{k}} \right] \;=\; (2\pi)^3\, \delta^3(\mathbf{p} - \mathbf{k})$$

Thus,

$$a_{\mathbf{p}}\, e^{i\Gamma}\; |0\rangle \;=\; \sum_{n=1}^{\infty} \frac{i^n}{n!}\, n\; [\, a_{\mathbf{p}}\, ,\, \Gamma\,]\; \Gamma^{n-1}\; |0\rangle \;=\; i\, f(\mathbf{p})\, e^{i\Gamma}\; |0\rangle$$

This leads to :

$$\langle f|\,\phi_+(x)\,|f\rangle \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{i}{\sqrt{2\,E_{\mathbf{p}}}}\,f(\mathbf{p})\,e^{-ip\cdot x}$$

Thus the expectation value of the field operator is :

$$\langle f|\,\phi(x)\,|f\rangle \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{i}{\sqrt{2\,E_{\mathbf{p}}}}\,\left(\,f(\mathbf{p})\,e^{-ip\cdot x}\;-\;f^*(\mathbf{p})\,e^{ip\cdot x}\,\right)$$

The expectation value trivially satisfies the Klein-Gordon equation since the field operator satisfies it and $|f\rangle$ is a Heisenberg state vector.

(b)

We evaluate the expectation value of the number density operator in momentum space for the coherent state $|f\rangle$:

$$\langle f|\;n_{\mathbf{p}}\;|f\rangle \;=\; \langle f|\;a^\dagger_{\mathbf{p}}\,a_{\mathbf{p}}\;|f\rangle \;=\; \left|N_f\,a_{\mathbf{p}}\,e^{i\Gamma}\;|0\rangle\right|^2 \;=\; |f(\mathbf{p})|^2$$

Also,

$$\begin{aligned}\langle f|\;n_{\mathbf{p}}\,n_{\mathbf{p}}\;|f\rangle \;&=\; |N_f|^2\;\langle 0|\;e^{-i\Gamma^\dagger}\,a^\dagger_{\mathbf{p}}\,a_{\mathbf{p}}\,a^\dagger_{\mathbf{p}}\,a_{\mathbf{p}}\,e^{i\Gamma}\;|0\rangle \\ &=\; |N_f|^2\;|f(\mathbf{p})|^2\;\langle 0|\;e^{-i\Gamma^\dagger}\,a_{\mathbf{p}}\,a^\dagger_{\mathbf{p}}\,e^{i\Gamma}\;|0\rangle\end{aligned}$$

From the commutation relations we see that this is a divergent quantity.

$$\langle f|\;n_{\mathbf{p}}\,n_{\mathbf{p}}\;|f\rangle \;=\; |f(\mathbf{p})|^2\;\left(|f(\mathbf{p})|^2\;+\;(2\pi)^3\,\delta^3(\mathbf{0})\right)$$

Thus,

$$\frac{\langle f|\;n_{\mathbf{p}}\,n_{\mathbf{p}}\;|f\rangle\;-\;(\langle f|\;n_{\mathbf{p}}\;|f\rangle)^2}{(\langle f|\;n_{\mathbf{p}}\;|f\rangle)^2}\;=\;\frac{(2\pi)^3\,\delta^3(\mathbf{0})}{|f(\mathbf{p})|^2}$$

This divergence arises since $n_{\mathbf{p}}$ is an operator-valued distribution and must be integrated before a well-defined product with another operator valued distribution may be taken.
We now evaluate the expectation value of the Hamiltonian for the coherent state $|f\rangle$:

$$\langle f|\;H\;|f\rangle \;=\; \int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\;\langle f|\;n_{\mathbf{p}}\;|f\rangle \;=\; \int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\;|f(\mathbf{p})|^2$$

Also,

$$\begin{aligned}\langle f|\;H^2\;|f\rangle \;&=\; \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\,E_{\mathbf{p}}\,E_{\mathbf{k}}\;\langle f|\;n_{\mathbf{p}}\,n_{\mathbf{k}}\;|f\rangle \\ &=\; |N_f|^2\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\,E_{\mathbf{p}}\,E_{\mathbf{k}}\;\langle 0|\;e^{-i\Gamma^\dagger}\,a^\dagger_{\mathbf{p}}\,a_{\mathbf{p}}\,a^\dagger_{\mathbf{k}}\,a_{\mathbf{k}}\,e^{i\Gamma}\;|0\rangle \\ &=\; |N_f|^2\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\,E_{\mathbf{p}}\,E_{\mathbf{k}}\,f^*(\mathbf{p})\,f(\mathbf{k})\;\langle 0|\;e^{-i\Gamma^\dagger}\,a_{\mathbf{p}}\,a^\dagger_{\mathbf{k}}\,e^{i\Gamma}\;|0\rangle\end{aligned}$$

Now,

$$\langle 0|\;e^{-i\Gamma^\dagger}\,a_{\mathbf{p}}\,a^\dagger_{\mathbf{k}}\,e^{i\Gamma}\;|0\rangle \;=\; |N_f|^{-2}\;\left(\,f(\mathbf{p})f^*(\mathbf{k})\;+\;(2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{k})\right)$$

Thus,

$$\langle f|\;H^2\;|f\rangle \;=\; \int \frac{d^3p}{(2\pi)^3}\,E^2_{\mathbf{p}}\;|f(\mathbf{p})|^2\;+\;\left(\int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\;|f(\mathbf{p})|^2\right)^2$$

Finally,

$$\frac{\langle f|\, H^2\, |f\rangle \,-\, (\langle f|\, H\, |f\rangle)^2}{(\langle f|\, H\, |f\rangle)^2} \;=\; \frac{\displaystyle\int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}}^2\, |f(\mathbf{p})|^2}{\left(\displaystyle\int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}}\, |f(\mathbf{p})|^2\right)^2}$$

(c)

As we found above, for (x, y) spacelike separated :

$$\langle 0|\, \phi(x)\phi(y)\, |0\rangle \;=\; D(x-y) \;\sim\; \frac{1}{(2\pi)^2\; r^2} \qquad \text{as} \qquad x \to y$$

Here, as above, $(x-y)^2 = -r^2$. Again, as for $n_{\mathbf{p}}$ above, this divergence arises since $\phi(x)$ is an operator-valued distribution. This is also a sign that local quantities that are quadratic in $\phi(x)$, such as the energy-momentum tensor, do not have well defined values and must be renormalized. Note that the divergence is independent of the mass of the particle. This is an indication that all particles behave as massless particles at sufficiently high energies (short distances).