QFT 1 : Problem Set 2

1.)

We begin by attempting to motivate the action for a relativistic point particle appearing in the homework set. Perhaps the more familiar action is that given by the invariant length of the worldline:

$$S_1\,[x] \;=\; -\,m \int ds \;(\,\eta_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu})^{1/2}$$

Where $\dot{x}^{\mu} \;\equiv\; \partial_s x^{\mu} \;\equiv\; \partial x^{\mu}/\partial s$.
In addition to being Poincare invariant, this action is invariant under arbitrary reparameterizations of the worldline coordinate s . The coordinates x^{μ} of course transform like scalars under this transformation. The equation of motion for $x(s)$ is the familiar:

$$\frac{\partial}{\partial s}\left(\,\dot{x}^{\nu}\;\,(\eta_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu})^{-1/2}\right) \;=\; 0$$

We now introduce a new action which incorporates a field which transforms like a metric under reparameterizations of the worldline coordinate s. We will see that it leads to the same equations of motion for $x(s)$.

$$S_2\,[x\,,\,g] \;=\; -\,\frac{m}{2}\int ds\,\sqrt{g_{ss}}\;(\,g^{ss}\,\partial_s x\,\cdot\,\partial_s x \;+\; 1\,)$$

Where $\partial_s x \cdot \partial_s x \;\equiv\; \eta_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu}$ and $g^{ss} g_{ss} = 1$. The equation of motion for g is found to be:

$$g_{ss} \;=\; \partial_s x \cdot \partial_s x$$

Thus, if the equations of motion are satisfied, g coincides with the metric along the worldline inherited (through its imbedding in spacetime) from η . From this we find:

$$S_2\,[\,x\,,\,\partial_s x\,\cdot\,\partial_s x\,] \;=\; S_1\,[x]$$

Thus the action S_2 leads to the same classical equations of motion for x as does S_1 . We now make a change of field variables by taking advantage of the fact that, in one dimension, we may replace the metric by a one-form. Thus we choose $N = \sqrt{g_{ss}}/m$. This leads to the action :

$$S_3\,[x\,,\,N] \;=\; -\,\frac{1}{2}\int ds\,\left(N^{-1}\,\dot{x}\cdot\dot{x} \;+\; N\,m^2\right)$$

From this action we derive the Hamiltonian:

$$\mathcal{H} \;=\; p_\mu \dot{x}^{\mu} \;-\; \mathcal{L} \;=\; -\,\tfrac{1}{2}\,N\left(p\cdot p \;-\; m^2\right)$$

Here $p_\mu = \partial\mathcal{L}/\partial\dot{x}^{\mu} \;=\; -\,N^{-1}\,\dot{x}_\mu$ and the canonical momentum of the N variable vanishes. We may define the path integral associated with S_3 as follows :

$$D_F(y-x) \;=\; \int \mathcal{D}N \int_{x(s_0)=x}^{x(s_1)=y} \mathcal{D}x\; e^{i\,S_3[x,N]}$$

Perhaps more fundamentally, we may consider the path integral to be defined through the use of the following action :

$$S_4\,[x\,,\,p\,,\,N] \;=\; \int ds\;(\,p_\mu\dot{x}^{\mu} \;-\; \mathcal{H}(p,x)\,) \;=\; \int ds\;\left(p_\mu\dot{x}^{\mu} \;+\; \tfrac{1}{2}\,N\left(p\cdot p \;-\; m^2\right)\right)$$

We write the path integral as :

$$D_F(y-x) \;=\; \int \mathcal{D}N \int \mathcal{D}p \int_{x(s_0)=x}^{x(s_1)=y} \mathcal{D}x \; e^{i\,S_4[x\,,\,p\,,\,N]}$$

This may be seen to give the same result as the path integral involving S_3 since the action S_4 is quadratic in p. Note that we do not integrate over the canonical momentum for N since it is identically zero. To ensure that this path integral converges we substitute $m^2 \to \big(m^2 - i\epsilon\big)$ and constrain the path integral to positive values of N .

In the discussion of gauge fixing we are going to diverge a bit from the statement of the problem. The problem presents a symmetry under canonical transformations induced by the Hamiltonian where N is treated as a Lagrange Multiplier for a first-class constraint. The symmetry is $\delta x = \alpha p\,,\; \delta p = 0\,,\; \delta N = -\partial_s \alpha$ for arbitrary $\alpha(s)$. I am more comfortable discussing fixing the symmetry of the action under diffeomorphisms; that is reparameterizations of the time parameter s . This symmetry treats x and p as scalars and N as a one-form so that $\delta x = -\beta \partial_s x\,,\; \delta p = -\beta \partial_s p\,,\; \delta N = -\partial_s(\beta N)$ for arbitrary $\beta(s)$. I strongly suspect that the symmetries are equivalent and certainly lead to the same result. This symmetry allows us to transform N subject to the condition that $\int ds\, N(s)$ is preserved as it must be under diffeomorphisms. The finite transformation of N is just the tensor transformation law:

$$\tilde{N}(\tilde{s})\,\frac{\partial \tilde{s}}{\partial s} \;=\; N(s)$$

We may use this freedom to transform any $N(s)$ to $\tilde{N}(\tilde{s}) = 1$. Then we have:

$$\int_{s_0}^{s_1} ds\, N(s) \;=\; \int_{\tilde{s}_0}^{\tilde{s}_1} d\tilde{s} \;=\; (\tilde{s}_1 \,-\, \tilde{s}_0) \;\equiv\; T$$

We may now do a further transformation to set the limits of the integral to $s_0 = 0$ and $s_1 = 1$ with $N(s) = T$. We cannot gauge away N entirely and T must be integrated over in the path integral. The principal reason for avoiding the tranformation in the homework is that I am not sure what the analog of the $\int ds\, N(s)$ constraint is. We are thus lead to the following gauge fixed path integral:

$$D_F(y-x) \;=\; \int_0^\infty dT \int \mathcal{D}p \int_{x(0)=x}^{x(1)=y} \mathcal{D}x \;\exp\left(i \int_0^1 ds \;\left[\, p\cdot\dot{x} \,+\, \tfrac{1}{2}\, T\,\left(p^2 - m^2 + i\epsilon\right) \right] \right)$$

(a)

We will respect the apparent time-honored tradition in theoretical physics of treating the solution of the path integral somewhat loosely. But first we present an expression that may be worked with to provide a perhaps more careful solution (here $\Delta = 1/n$) :

$$D_F(y-x) \;=\; \lim_{n\to\infty} \quad \int_0^\infty dT \left(\prod_{j=0}^{n} \int d^d x_j \right) \left(\prod_{k=1}^{n} \int \frac{d^d p_k}{(2\pi)^d} \right) \delta^d(x_0 - x)\, \delta^d(x_n - y)$$
$$\times \exp\left(i\,\Delta \sum_{k=1}^{n} \left[\, p_k \cdot (x_k - x_{k-1})\,/\Delta \,-\, T/2\,\left(p_k\cdot p_k - m^2 + i\epsilon\right) \right] \right)$$

We treat the x path integral, following an integration by parts in the action, as a functional Fourier transform :

$$\int_{x(0)=x}^{x(1)=y} \mathcal{D}x \;\exp\left(i \int_0^1 ds \;(p\cdot\dot{x}) \right) \;=\; \delta\,[\,\dot{p}\,]\;\exp\left(\, i\,\left[\,p(1)\cdot y \,-\, p(0)\cdot x\,\right]\right)$$

Inserting this into the path integral we blithely convert the p functional integral into an ordinary integral since $\dot{p} = 0$:

$$D_F(y-x) \;=\; \lambda_d \int_0^\infty dT \int \frac{d^d p}{(2\pi)^d} \, \exp\left(i \left[p \cdot (y-x) + \tfrac{1}{2} T \left(p^2 - m^2 + i\epsilon \right) \right] \right)$$

Note that the integral over s in the action produces its integrand since $\dot{p} = 0$. Also note that a factor λ_d has been inserted to provide the normalization to be determined below. We now use the following formula for the integral of a gaussian :

$$\int d^d p \, e^{-i\, p\cdot\alpha} \, e^{-a\, p^2} \;=\; e^{-\alpha^2/(4a)} \left(\frac{\pi}{a} \right)^{d/2}$$

Setting $a = -i\,T/2$ and $\alpha = (x-y)$ we find :

$$D_F(y-x) \;=\; \lambda_d \left(\frac{i}{2\pi} \right)^{d/2} \int_0^\infty dT \, T^{-d/2} \exp\left(-i/2 \left[T \left(m^2 - i\epsilon \right) + T^{-1} \left(y - x \right)^2 \right] \right)$$

(b)

We now show that D_F is a Green function for the Klein-Gordon equation. We find that :

$$\begin{aligned}
&\left(\partial^2 + m^2 - i\epsilon \right) \, D_F(x) \\
&\quad = \lambda_d \left(\frac{i}{2\pi} \right)^{d/2} \int_0^\infty dT \, T^{-d/2} \left(m^2 - i\epsilon - x^2/T^2 - id/T \right) \\
&\qquad\qquad \times \exp\left(-i/2 \left[T \left(m^2 - i\epsilon \right) + T^{-1} x^2 \right] \right) \\
&\quad = 2i\,\lambda_d \left(\frac{i}{2\pi} \right)^{d/2} \int_0^\infty dT \, \frac{d}{dT} \left(T^{-d/2} \exp\left(-i/2 \left[T \left(m^2 - i\epsilon \right) + T^{-1} x^2 \right] \right) \right) \\
&\quad = -2i\,\lambda_d \left(\frac{i}{2\pi} \right)^{d/2} \lim_{T\to 0} \left[T^{-d/2} \exp\left(-i\, x^2 / \left(2\,T \right) \right) \right] \;=\; -2i\,\lambda_d \, \delta^d(x)
\end{aligned}$$

To see that the distribution given here is a delta function we may integrate it against a test function. We will find that the integral oscillates wildly in the $T \to 0$ limit except near $x = 0$. We remove the test function (evaluated at $x = 0$) and the distribution integrates to 1 since it is a normalized gaussian for all T . This is of course pretty loose language but is essentially correct. To verify this result we return to the (normalized) expression for D_F prior to performing the momentum integral. To conform with the definition in Peskin and Schroeder we choose the normalization $\lambda_d = 1/2$ and take $p \to -p$ in the integral :

$$D_F(x) \;=\; \frac{1}{2} \int_0^\infty dT \int \frac{d^d p}{(2\pi)^d} \, \exp\left(-i \left[p \cdot x - \tfrac{1}{2} T \left(p^2 - m^2 + i\epsilon \right) \right] \right)$$

We perform the T integral to find ;

$$D_F(x) \;=\; \int \frac{d^d p}{(2\pi)^d} \, \frac{i\, e^{-i\, p\cdot x}}{(p^2 - m^2 + i\epsilon)}$$

Thus,

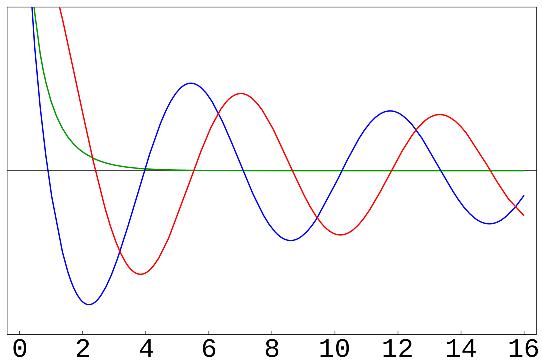
$$\left(\partial^2 + m^2 - i\epsilon \right) \, D_F(x) \;=\; -i\, \delta^d(x)$$

(c)

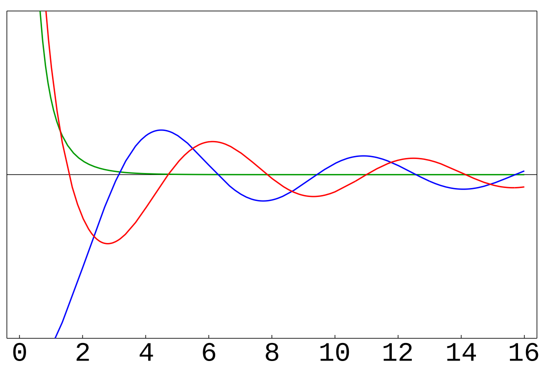
The most straightforward way to approach this problem is to use a table of integrals or plug the expression for D_F as an integral over T into a program like mathematica. The result is:

$$D_F(x) \;=\; \frac{1}{2\pi}\left(\frac{i\,m}{2\pi\,\sqrt{x^2}}\right)^{(d/2-1)} K_{(d/2-1)}(\,-i\,m\,\sqrt{x^2})$$

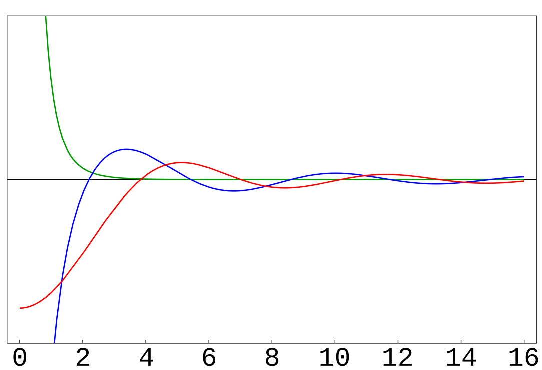
The following are graphs of $x^{(1-d/2)}\,K_{d/2-1}(x)$ (spacelike) in green and the real and imaginary parts of $(-ix)^{(1-d/2)}\,K_{d/2-1}(-ix)$ (timelike) in blue and red respectively.
For $d = 2$:



For $d = 3$:



For $d = 4$:



2.) Peskin & Schroeder 9.2 (a-c)

(a)

We want to express the quantum statistical partition function in terms of a functional integral. For notational clarity we will consider a one-dimensional single particle quantum system. The extension to a more complicated system is trivial. Inserting a complete set of position eigenstates we have:

$$Z(\beta) \;=\; \mathrm{tr}\left[e^{-\beta\, H}\right] \;=\; \int dx\; \langle x|\, e^{-\beta H}\, |x\rangle$$

Rather than evaluating the propagator for complex time and facing relatively delicate issues related to analytic continuation, we derive the path integral directly. Defining $\epsilon = \beta/N$ and inserting complete sets of position and momentum eigenstates, we have:

$$Z(\beta) \;=\; \lim_{N\to\infty} \int dx \left(\prod_{j=0}^{N} \int dx_j\right) \left(\prod_{k=1}^{N} \int dp_k\right)$$

$$\times\; \delta(x_N - x)\, \delta(x_0 - x) \prod_{n=1}^{N} \left[\, \langle x_n|\, e^{-\epsilon H}\, |p_n\rangle\; \langle p_n\, |x_{n-1}\rangle \,\right]$$

Now,

$$\prod_{n=1}^{N} \left(\langle x_n\, |p_n\rangle\; \langle p_n|\, e^{-\epsilon H}\, |x_{n-1}\rangle\right) \;=\; \exp\left[-\epsilon \sum_{n=1}^{N} \left(\, -ip_n\,(x_n - x_{n-1})\,/\epsilon \,+\, H\,(p_n, x_n)\,\right)\right]$$

Thus, using the definition of the phase space path integral appearing in P&S, we may write the partition function as:

$$Z(\beta) \;=\; \int dx \int_{q(0)=x}^{q(\beta)=x} \mathcal{D}q \int \mathcal{D}p\; \exp\left[-\int_0^\beta ds\, \tilde{L}_E(q, \dot{q}, p)\right]$$

Where we are using the hybrid Euclidean Lagrangian:

$$\tilde{L}_E(q, \dot{q}, p) \;=\; -i\, p\, \dot{q} \,+\, H(p, q)$$

Note that the path integral is over all periodic paths that have period β . If $H(p, q)$ can be written as $H = p^2/2m + V(q)$, we may evaluate the p integrals explicitly:

$$Z(\beta) \;=\; \lim_{N\to\infty} \int dx \left(\prod_{j=0}^{N} \int dx_j\right) \delta(x_N - x)\, \delta(x_0 - x)\; \exp\left[-\epsilon \sum_{n=1}^{N} V\,(x_n)\right]$$

$$\times \left(\prod_{k=1}^{N} \int dp_k\right) \exp\left[-\epsilon \sum_{n=1}^{N} \left(\, -ip_n\,(x_n - x_{n-1})\,/\epsilon \,+\, p_n^2/2m\,\right)\right]$$

Thus,

$$Z(\beta) \;=\; \lim_{N\to\infty} \int dx \left(\prod_{j=0}^{N} \int dx_j\right) \left(\frac{m}{2\pi\epsilon}\right)^{N/2} \delta(x_N - x)\, \delta(x_0 - x)$$

$$\times\; \exp\left[-\epsilon \sum_{n=1}^{N} \left(\, \frac{m}{2}\,(x_n - x_{n-1})^2\,/\epsilon^2 \,+\, V\,(x_n)\right)\right]$$

Again, using the definition of the configuration space path integral appearing in P&S, we may write the partition function as:

$$Z(\beta) \;=\; \int dx \int_{q(0)=x}^{q(\beta)=x} \tilde{\mathcal{D}}q \;\exp\left[-\int_0^\beta ds\, L_E(q,\dot{q})\right]$$

Where we are using the Euclidean Lagrangian:

$$L_E(q,\dot{q}) \;=\; \frac{m}{2}\,\dot{q}^2 \;+\; V(q)$$

The measure in the configuration space path integral is written as $\tilde{\mathcal{D}}q$ to reflect the additional factors in the measure that do not appear in the phase space path integral.

(b)

We consider the Euclidean action for the unit mass harmonic oscillator:

$$\mathcal{L} \;=\; \tfrac{1}{2}\left(\dot{x}^2 \;+\; \omega^2 x^2\right)$$

Since we are considering a path integral over periodic functions, we expand $x(t)$ in a Fourier series:

$$x(t) \;=\; \frac{1}{\sqrt{\beta}}\sum_{n\in\mathbb{Z}} x_n\, e^{2\pi i n t/\beta} \qquad \text{and} \qquad x_n \;=\; \frac{1}{\sqrt{\beta}}\int_0^\beta dt\, x(t)\, e^{-2\pi i n t/\beta}$$

The reality of $x(t)$ imposes $x_n^* \;=\; x_{-n}$. We will proceed in a cavalier manner and simply define the path integral measure to be:

$$\int dx \int_{q(0)=x}^{q(\beta)=x} \tilde{\mathcal{D}}q \;=\; \prod_{n=0}^{\infty}\prod_{m=1}^{\infty}\int da_n db_m$$

Where $x_n \;=\; a_n + i b_n$. Note that b_0 is absent due to the reality condition. We will pay dearly below for this choice of measure in the form of infinite β-dependent prefactors. It is possible to avoid these infinities through a more careful definition of the discrete form of the path integral (see Itzykson and Zuber 9-1). We will proceed as P&S intends and neglect the divergent pieces. We first compute the action in terms of the Fourier modes.

$$S_E \;=\; \int_0^\beta dt\, \tfrac{1}{2}\left(\dot{x}^2 \;+\; \omega^2 x^2\right)$$

Now,

$$\begin{aligned}
\int_0^\beta dt\, x^2(t) \;&=\; \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \frac{1}{\beta}\, x_n\, x_m \int_0^\beta dt\, e^{2\pi i (n+m)t/\beta} \\
&=\; \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_n\, x_m\, \delta(m+n) \;=\; a_0^2 \;+\; 2\sum_{n=1}^{\infty}\left(a_n^2 \;+\; b_n^2\right)
\end{aligned}$$

And,

$$\begin{aligned}
\int_0^\beta dt\, \dot{x}^2(t) \;&=\; \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \frac{1}{\beta}\,(-mn)\left(\frac{2\pi}{\beta}\right)^2 x_n\, x_m \int_0^\beta dt\, e^{2\pi i (n+m)t/\beta} \\
&=\; \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} (-mn)\left(\frac{2\pi}{\beta}\right)^2 x_n\, x_m\, \delta(m+n) \;=\; 2\left(\frac{2\pi}{\beta}\right)^2 \sum_{n=1}^{\infty} n^2\left(a_n^2 \;+\; b_n^2\right)
\end{aligned}$$

Thus,

$$S_E \;=\; \tfrac{1}{2}\omega^2 a_0^2 \,+\, \sum_{n=1}^{\infty}\left(\left(\frac{2\pi n}{\beta}\right)^2 \,+\, \omega^2\right)\left(\, a_n^2 \,+\, b_n^2\,\right)$$

This leads to:

$$Z(\beta) \;=\; \int da_0\, e^{-\frac{1}{2}\omega^2 a_0^2}\,\prod_{n=1}^{\infty}\prod_{m=1}^{\infty}\int\, da_n db_m\; e^{-\left(\left(\frac{2\pi n}{\beta}\right)^2 + \omega^2\right)\left(\, a_n^2 + b_n^2\,\right)}$$

Or,

$$Z(\beta) \;=\; \left(\frac{2\pi}{\omega^2}\right)^{1/2}\,\prod_{n=1}^{\infty}\pi\left(\left(\frac{2\pi n}{\beta}\right)^2 \,+\, \omega^2\right)^{-1}$$

We may write this as :

$$Z(\beta) \;=\; \left[\,2\,\beta\,\sqrt{\frac{\pi}{2}}\,\prod_{n=1}^{\infty}\pi\left(\frac{2\pi n}{\beta}\right)^2\,\right]\,(\beta\omega)^{-1}\,\prod_{n=1}^{\infty}\left(1 \,+\, \frac{(\beta\omega/2)^2}{(\pi n)^2}\right)^{-1}$$

Neglecting the ω-independent factor in backets and using the product representation for sinh appearing in P&S we find:

$$Z(\beta) \;=\; \left(\,2\,\sinh\left(\beta\omega/2\right)\,\right)^{-1}$$

You are invited to feel troubled by this derivation.

(c)

We formulate the partition function for a real scalar field by first considering the following matrix element.

$$U(\phi_a,\phi_b|-i\gamma) \;\equiv\; \langle\,\phi_b\,|\;e^{-\gamma H}\;|\,\phi_a\,\rangle$$

We are working in the Schroedinger picture with Hamiltonian:

$$H\,[\,\pi\,,\,\phi\,] \;=\; \int d^3x\,\mathcal{H}(\pi,\nabla\phi,\phi) \;=\; \frac{1}{2}\int d^3x\left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right)$$

Rather than treating the problem of real and imaginary time separately, with time or temperature as a continuous parameter in the path integral, it is more straightforward to introduce a continuous parameter which indexes the insertion of an infinite number of complete sets of states. Defining $\epsilon = 1/N$ and inserting complete sets of field and momentum eigenstates:

$$U(\phi_a,\phi_b|\,i\gamma) \;=\; \lim_{N\to\infty}\left(\prod_{j=0}^{N}\int\mathcal{D}\phi_j(\mathbf{x}_j)\right)\left(\prod_{k=1}^{N}\int\mathcal{D}\pi_k(\mathbf{x}_k)\right)$$

$$\times\,\delta\left[\phi_0-\phi_a\right]\,\delta\left[\phi_N-\phi_b\right]\,\prod_{n=1}^{N}\,\left[\,\langle\phi_n|\,e^{-\epsilon\,\gamma H}\,|\pi_n\rangle\;\langle\pi_n\,|\phi_{n-1}\rangle\,\right]$$

Where we have introduced the functional delta function:

$$\mathcal{F}[\tilde{\phi}] \;=\; \int\mathcal{D}\phi\;\delta[\phi-\tilde{\phi}]\;\mathcal{F}[\phi]$$

From the canonical commutation relations:

$$\exp\left[\,-i\Gamma\left[\,\pi\,,\,\phi\,\right]\,\right] \;\equiv\; \langle\pi\,|\phi\rangle \;=\; \exp\left[-i\int d^3x\,\pi(\mathbf{x})\,\phi(\mathbf{x})\right]$$

Thus,

$$\prod_{n=1}^{N} \left[\langle \phi_n | \, e^{-\epsilon\,\gamma H} \, |\pi_n\rangle \, \langle \pi_n \, |\phi_{n-1}\rangle \, \right]$$

$$= \; \exp \left[\epsilon \sum_{n=1}^{N} \left[\, i\Gamma \left[\pi_n, (\phi_n - \phi_{n-1}) \, / \epsilon \, \right] \, - \, \gamma H \left[\pi_n, \phi_n \right] \right] \right]$$

We now introduce a continuous parameter σ which indexes the complete set of states and define a hybrid Lagrangian:

$$\tilde{L}_\gamma[\dot{\phi}, \phi, \pi] \; = \; \Gamma[\pi, \dot{\phi}] \, + \, i \, \gamma H \left[\pi, \phi \right] \qquad \text{where} \qquad \dot{\phi} \, \equiv \, \frac{\partial \phi}{\partial \sigma}$$

This leads to the path integral form of the matrix element:

$$U(\phi_a, \phi_b | - i\gamma) \; = \; \int_{\phi(0)=\phi_a}^{\phi(1)=\phi_b} \mathcal{D}\phi(\sigma) \int \, \mathcal{D}\pi(\sigma) \; \exp \left[i \int_0^1 d\sigma \; \tilde{L}_\gamma[\dot{\phi}, \phi, \pi] \right]$$

Substituting $\gamma = it$ and $\phi(\sigma) \to \phi(\sigma t)$ and changing variables to $s = \sigma t$ we find:

$$U(\phi_a, \phi_b | \, t) \; = \; \int_{\phi(0)=\phi_a}^{\phi(t)=\phi_b} \mathcal{D}\phi(s) \int \, \mathcal{D}\pi(s) \; \exp \left[i \int_0^t ds \, \int d^3x \, \tilde{\mathcal{L}}(\dot{\phi}, \nabla\phi, \phi, \pi) \right]$$

Where,

$$\tilde{\mathcal{L}}(\dot{\phi}, \nabla\phi, \phi, \pi) \; = \; \pi\dot{\phi} \, - \, \mathcal{H}(\pi, \nabla\phi, \phi) \qquad \text{where} \qquad \dot{\phi} \, \equiv \, \frac{\partial \phi}{\partial s}$$

The partition function is defined as:

$$Z(\beta) \; = \; \int \mathcal{D}\phi_0 \; U(\phi_0, \phi_0 | - i\beta)$$

Substituting $\gamma = \beta$ and $\phi(\sigma) \to \phi(\sigma\beta)$ and changing variables to $s = \sigma\beta$ we find:

$$Z(\beta) \; = \; \int \mathcal{D}\phi_0 \, \int_{\phi(0)=\phi_0}^{\phi(\beta)=\phi_0} \mathcal{D}\phi(s) \int \, \mathcal{D}\pi(s) \; \exp \left[- \int_0^\beta ds \, \int d^3x \, \tilde{\mathcal{L}}_E(\dot{\phi}, \nabla\phi, \phi, \pi) \right]$$

Where,

$$\tilde{\mathcal{L}}_E(\dot{\phi}, \nabla\phi, \phi, \pi) \; = \; -i \, \pi\dot{\phi} \, + \, \mathcal{H}(\pi, \nabla\phi, \phi) \qquad \text{where} \qquad \dot{\phi} \, \equiv \, \frac{\partial \phi}{\partial s}$$

We now take advantage of the fact that the Hamiltonian is quadratic in π and integrate out the momentum variables. We write the path integral as:

$$U(\phi_a, \phi_b | - i\gamma) \; = \; \int_{\phi(0)=\phi_a}^{\phi(1)=\phi_b} \mathcal{D}\phi(\sigma) \exp \left[- \frac{\gamma}{2} \int_0^1 d\sigma \, \int d^3x \left(\, (\nabla\phi)^2 + m^2\phi^2 \, \right) \right]$$

$$\times \int \mathcal{D}\pi(\sigma) \; \exp \left[- \int_0^1 d\sigma \, \int d^3x \, \left(\frac{\gamma}{2} \, \pi^2 \, - \, i \, \pi \, \dot{\phi} \right) \right]$$

Performing the gaussian integral and absorbing γ dependent terms into the measure:

$$U(\phi_a, \phi_b | - i\gamma) \; = \; \int_{\phi(0)=\phi_a}^{\phi(1)=\phi_b} \tilde{\mathcal{D}}\phi(\sigma) \exp \left[- \frac{\gamma}{2} \int_0^1 d\sigma \, \int d^3x \left(\, \gamma^{-2} \dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2 \, \right) \right]$$

Substituting $\gamma = it$ and $\phi(\sigma) \to \phi(\sigma t)$ and changing variables to $s = \sigma t$ we find:

$$U(\phi_a, \phi_b | \, t) \; = \; \int_{\phi(0)=\phi_a}^{\phi(t)=\phi_b} \tilde{\mathcal{D}}\phi(s) \; \exp\left[\, i \int_0^t ds \int d^3x \, \mathcal{L}(\dot{\phi}, \nabla\phi, \phi) \right]$$

Where,

$$\mathcal{L}(\dot{\phi}, \nabla\phi, \phi) \; = \; \frac{1}{2} \left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2 \right) \qquad \text{where} \qquad \dot{\phi} \; \equiv \; \frac{\partial\phi}{\partial s}$$

Substituting $\gamma = \beta$ and $\phi(\sigma) \to \phi(\sigma\beta)$ and changing variables to $s = \sigma\beta$ we find:

$$Z(\beta) \; = \; \int \mathcal{D}\phi_0 \int_{\phi(0)=\phi_0}^{\phi(\beta)=\phi_0} \tilde{\mathcal{D}}\phi(s) \; \exp\left[- \int_0^\beta ds \int d^3x \, \mathcal{L}_E(\dot{\phi}, \nabla\phi, \phi) \right]$$

Where,

$$\mathcal{L}_E(\dot{\phi}, \nabla\phi, \phi) \; = \; \frac{1}{2} \left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2 \right) \qquad \text{where} \qquad \dot{\phi} \; \equiv \; \frac{\partial\phi}{\partial s}$$

Integrating by parts we have:

$$Z(\beta) \; = \; \int \mathcal{D}\phi_0 \int_{\phi(0)=\phi_0}^{\phi(\beta)=\phi_0} \tilde{\mathcal{D}}\phi(s) \; \exp\left[-\frac{1}{2} \int_0^\beta ds \int d^3x \, \phi \left(-\partial_s^2 - \nabla^2 + m^2 \right) \phi \right]$$

Or,

$$Z(\beta) \; = \; \left(\det(-\partial_E^2 + m^2) \right)^{-1/2}$$

We will compute this path integral in a manner analogous to that used for the partition function for the harmonic oscillator. We introduce periodic boundary conditions on $\mathbb{R}^3$ and Fourier decompose $\phi(\mathbf{x}, s)$ ($V = L^3$) :

$$\phi(\mathbf{x}, s) \; = \; \frac{1}{\sqrt{\beta V}} \sum_{n \in \mathbb{Z}} \sum_{\mathbf{n} \in \mathbb{Z}^3} \phi_{(n,\mathbf{n})} \, e^{2\pi i \, ns/\beta} \, e^{2\pi i \, \mathbf{n} \cdot \mathbf{x}/L}$$

Since ϕ is real, if we define $\phi_{(n,\mathbf{n})} = A_{(n,\mathbf{n})} + iB_{(n,\mathbf{n})}$, we find :

$$A_{(n,\mathbf{n})} \; = \; A_{(-n,-\mathbf{n})} \qquad \text{and} \qquad B_{(n,\mathbf{n})} \; = \; -B_{(-n,-\mathbf{n})}$$

This allows us to define the functional measure as:

$$\int \mathcal{D}\phi_0 \int_{\phi(0)=\phi_0}^{\phi(\beta)=\phi_0} \tilde{\mathcal{D}}\phi(s) \; = \; \left[\prod_{n \geq 0} \prod_{\mathbf{n}} \int dA_{(n,\mathbf{n})} \right] \left[\prod_{m > 0} \prod_{\mathbf{m}} \int dB_{(m,\mathbf{m})} \right]$$

With some algebra we find:

$$\frac{1}{2} \int_0^\beta ds \int d^3x \, \phi \left(-\partial_s^2 - \nabla^2 + m^2 \right) \phi \; = \; \sum_{\mathbf{n}} \frac{1}{2} A_{(0,\mathbf{n})}^2 \left(m^2 + (2\pi\mathbf{n}/L)^2 \right)$$

$$+ \sum_{\mathbf{n}} \sum_{n > 0} \left(A_{(n,\mathbf{n})}^2 + B_{(n,\mathbf{n})}^2 \right) \left(m^2 + (2\pi n/\beta)^2 + (2\pi\mathbf{n}/L)^2 \right)$$

Defining,

$$I_{(n,\mathbf{n})} \; = \; \int d\alpha \, \exp\left[-\alpha^2 \, \left(m^2 + (2\pi n/\beta)^2 + (2\pi\mathbf{n}/L)^2 \right) \right]$$

We have:

$$Z(\beta) \;=\; \prod_{\mathbf{n}} \left[\sqrt{2}\, I(0,\mathbf{n}) \prod_{n>0} (\, I(n,\mathbf{n})\,)^2 \right]$$

Defining,

$$\omega_{\mathbf{n}}^2 \;=\; \left(m^2 + (2\pi\mathbf{n}/L)^2 \,\right)$$

We find:

$$I_{(n,\mathbf{n})} \;=\; \sqrt{\pi}\, \left((2\pi n/\beta)^2 + \omega_{\mathbf{n}}^2 \,\right)^{-1/2}$$

Thus,

$$Z(\beta) \;=\; \prod_{\mathbf{n}} \left[\frac{\sqrt{2\pi}}{\omega_{\mathbf{n}}} \prod_{n>0} \pi \left((2\pi n/\beta)^2 + \omega_{\mathbf{n}}^2 \,\right)^{-1} \right]$$

Dropping $\omega_{\mathbf{n}}$-independent factors as in part (**b**) above, we may write this as :

$$Z(\beta) \;=\; \prod_{\mathbf{n}} (\, 2 \sinh(\beta\, \omega_{\mathbf{n}}/2)\,)^{-1}$$

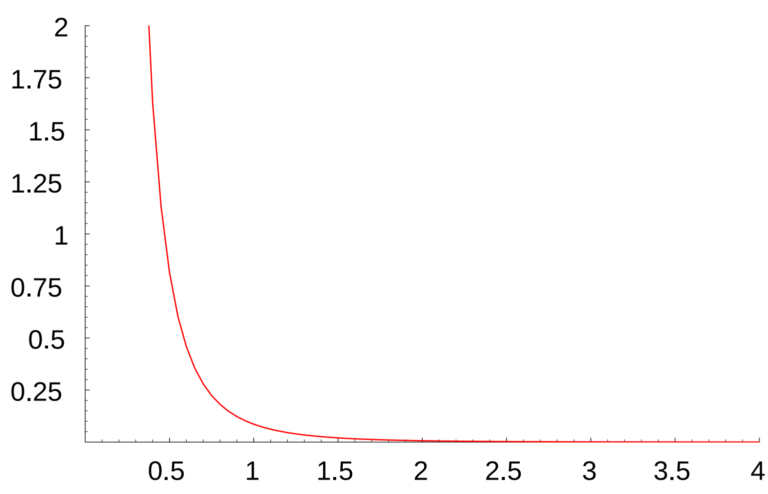
We rewrite this as:

$$\ln Z(\beta) \;=\; -\sum_{\mathbf{n}} (\beta\, \omega_{\mathbf{n}}/2) \;-\sum_{\mathbf{n}} \ln (\, 1 - \exp(-\beta\, \omega_{\mathbf{n}})\,)$$

Dropping the first term, which amounts to the normal ordering prescription, and writing the sum as an integral over $\mathbf{k}$ we find:

$$\ln Z(\beta) \;=\; -V \int \frac{d^3k}{(2\pi)^3}\, \ln \left(1 - \exp\left(-\beta \sqrt{\mathbf{k}^2 + m^2} \right) \right)$$

As for the harmonic oscillator, this derivation is much simpler using operator methods. Please see Itzykson and Zuber 3-1-5. The following is a plot of $\ln Z(\beta)/(Vm^3)$ as a function of βm obtained through numerical integration.



3.)

We begin with the following Lagrangian:

$$\mathcal{L}_0 \;=\; \psi^* \dot{\psi} \;-\; \frac{1}{2m}\, \nabla \psi^* \cdot \nabla \psi$$

This leads to the Schroedinger equation as the wave equation obeyed by the field (that is, the creation/annihilation operators will create/destroy particles in Schroedinger wave modes):

$$i\, \dot{\psi} \;=\; -\frac{1}{2m}\, \nabla^2 \psi$$

We find that the canonical momenta are:

$$\pi_\psi \;\equiv\; \frac{\partial \mathcal{L}_0}{\partial \dot{\psi}} \;=\; i\, \psi^* \qquad \text{and} \qquad \pi_{\psi^*} \;\equiv\; \frac{\partial \mathcal{L}_0}{\partial \dot{\psi}^*} \;=\; 0$$

The Hamiltonian (density) is:

$$\mathcal{H}_0 \;\equiv\; \pi_\psi \dot{\psi} + \pi_{\psi^*} \dot{\psi}^* - \mathcal{L}_0 \;=\; \frac{1}{2m}\, \nabla \psi^* \cdot \nabla \psi \;=\; \frac{-i}{2m}\, \nabla \pi \cdot \nabla \psi$$

Essentially, because the Lagrangian is linear in $\dot{\psi}$, the phase space for each eigenmode of the wave operator (labelled by the spatial momentum $\mathbf{p}$) is two dimensional rather than four dimensional. We could have placed ψ and ψ^* on a more equal footing but this would have required us to implement constraints to define the Hamiltonian. In what follows we treat ψ and its momentum π as the only phase space variables. Note that the equation of motion for ψ^* follows from the Hamilton equation for $\dot{\pi}$. This is similar to how the Dirac equation is generally treated in the Hamiltonian formalism.
We now implement the canonical commutation relations:

$$[\, \psi(\mathbf{x}, t)\, ,\, \pi(\mathbf{y}, t)\,] \;=\; i\, \delta^3(\mathbf{x} - \mathbf{y})$$

These may be implemented by defining ψ as follows:

$$\psi(\mathbf{x}, t) \;=\; \int \frac{d^3p}{(2\pi)^3}\, a_{\mathbf{p}}\, e^{i\mathbf{p}\cdot\mathbf{x} - i\,\omega(\mathbf{p})\, t} \qquad \text{where} \qquad \omega(\mathbf{p}) \;=\; \frac{\mathbf{p}^2}{2m}$$

where, as usual,

$$\left[a_{\mathbf{p}}\, ,\, a^\dagger_{\mathbf{k}} \right] \;=\; (2\pi)^3\, \delta^3(\mathbf{p} - \mathbf{k})$$

Note that one does not need to include a term $b^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x} + i\omega(\mathbf{p})t}$ in the mode expansion, as there are no negative frequency solutions to the nonrelativistic dispersion relation $\omega = \mathbf{p}^2/2m$, in contrast to the relativistic version $\omega = \pm\sqrt{\mathbf{p}^2 + m^2}$. The time dependence of this choice is consistent with the Hamiltonian:

$$H_0 \;=\; \int d^3x\, \mathcal{H}_0 \;=\; \int d^3x\, \frac{1}{2m}\, \nabla \psi^* \cdot \nabla \psi \;=\; \int \frac{d^3p}{(2\pi)^3}\, \frac{\mathbf{p}^2}{2m}\, a^\dagger_{\mathbf{p}}\, a_{\mathbf{p}}$$

We now introduce an interaction of the form:

$$H_1(t) \;=\; \frac{1}{2} \int d^3x \int d^3y\; |\psi(\mathbf{x}, t)|^2\;\, U(\mathbf{x} - \mathbf{y})\;\, |\psi(\mathbf{y}, t)|^2$$

This is exactly the same expression as the interaction energy of two particles in nonrelativistic QM interacting via the potential $U(\mathbf{x} - \mathbf{y})$. The only difference is that ψ is a quantum field rather than a wavefunction.[1] Nevertheless, it serves much the same purpose, since $\psi^\dagger\psi$ is the operator that measures the particle density. Note that, in the Heisenberg picture, this operator is not time independent since it does not commute with H_0. However, importantly, it may be shown that it does commute with the number operator:

$$N \;=\; \int \frac{d^3p}{(2\pi)^3}\, a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$$

Thus, as expected, the number of particles is conserved in the non-relativistic interaction. To see that this interaction is the correct one we consider an n-particle state consisting of position eigenfunctions localized at time $t = 0$:

$$|\, x_1 \ldots x_n \,\rangle \;=\; \psi^\dagger(\mathbf{x}_1, 0)\, \ldots\, \psi^\dagger(\mathbf{x}_n, 0)\; |0\rangle \;=\; \left[\prod_{j=1}^{n} \int \frac{d^3p_j}{(2\pi)^3}\, e^{-i\mathbf{p}_j\cdot\mathbf{x}_j}\, a^\dagger_{\mathbf{p}_j} \right] |0\rangle$$

Denoting $\psi_{\mathbf{x}} = \psi(\mathbf{x}, 0)$, we compute:

$$: H_1(0) :\; |\, x_1 \ldots x_n \,\rangle \;=\; \frac{1}{2} \int d^3x \int d^3y\, \psi^\dagger_{\mathbf{x}}\, \psi^\dagger_{\mathbf{y}}\, U(\mathbf{x}-\mathbf{y})\, \psi_{\mathbf{x}}\, \psi_{\mathbf{y}}\, \psi^\dagger_{\mathbf{x}_1}\, \ldots\, \psi^\dagger_{\mathbf{x}_n}\; |0\rangle$$

Using the canonical commutation relations we find:

$$: H_1(0) :\; |\, x_1 \ldots x_n \,\rangle \;=\; \frac{1}{2} \sum_{j=1}^{n} \sum_{k\neq j}^{n} U(\mathbf{x}_k - \mathbf{y}_j)\; |\, x_1 \ldots x_n \,\rangle$$

Note that the normal ordering removes the self-interactions. We now consider the system in a large cubic box of volume $V = L^3$. We replace the field expansion ($t = 0$) by :

$$\psi_{\mathbf{x}} \;=\; \frac{1}{V} \sum_{\mathbf{n}\in\mathbb{Z}} a_{\mathbf{p}}\, e^{i\mathbf{p}\cdot\mathbf{x}} \qquad \text{where} \qquad \mathbf{p} \;=\; \frac{2\pi\mathbf{n}}{L}$$

And the commutation relations are:

$$\left[\, \psi_{\mathbf{x}} \,,\, \psi^\dagger_{\mathbf{y}} \,\right] \;=\; \delta^3(\mathbf{x}-\mathbf{y}) \qquad \text{and} \qquad \left[\, a_{\mathbf{p}} \,,\, a^\dagger_{\mathbf{k}} \,\right] \;=\; V\, \delta_{\mathbf{pk}}$$

We now compute the expectation value of the non-interacting N-particle ground state $|\Omega_N\rangle$ to first order in perturbation theory. That is we compute :

$$E^{(1)} \;=\; \langle \Omega_N |\;\, : H_1(0) :\; |\Omega_N\rangle$$

Where,

$$|\Omega_N\rangle \;=\; \frac{V^{-N/2}}{\sqrt{N!}} \left(a^\dagger_{\mathbf{0}} \right)^N \; |0\rangle$$

Thus,

$$E^{(1)} \;=\; \frac{V^{-N}}{N!}\, \frac{1}{2} \int d^3x \int d^3y\, U(\mathbf{x}-\mathbf{y})\; \langle 0 | \left(a_{\mathbf{0}} \right)^N \psi^\dagger_{\mathbf{x}}\, \psi^\dagger_{\mathbf{y}} \psi_{\mathbf{x}}\, \psi_{\mathbf{y}} \left(a^\dagger_{\mathbf{0}} \right)^N \; |0\rangle$$

[1] By the way, the terminology "second quantization" refers to this distinction – ordinary quantum mechanics ("first quantization") has $\mathbf{x}$ as the dynamical variable, and ψ as the wavefunction. Here we have ψ as the dynamical variable, and the states are formally described by wavefunctions $\Psi(\psi)$. In a garbled sense, one is quantizing the wavefunction, hence the terminology "second quantization". But this abuse of terminology arises from overlooking the fact that in field theory, the field always was the dynamical object, not $\mathbf{x}$ which simply parametrizes the space on which the field is defined.

Now,

$$\psi_{\mathbf{x}}\,\psi_{\mathbf{y}}\left(a^\dagger_{\mathbf{0}}\right)^N|0\rangle \;=\; N\,\psi_{\mathbf{x}}\left(a^\dagger_{\mathbf{0}}\right)^{N-1}|0\rangle \;=\; N\;(N-1)\;\left(a^\dagger_{\mathbf{0}}\right)^{N-2}|0\rangle$$

Thus,

$$\langle 0|\left(a_{\mathbf{0}}\right)^N\psi^\dagger_{\mathbf{x}}\,\psi^\dagger_{\mathbf{y}}\psi_{\mathbf{x}}\,\psi_{\mathbf{y}}\left(a^\dagger_{\mathbf{0}}\right)^N|0\rangle \;=\; N\,(N-1)\;N!\,V^{N-2}$$

Also,

$$\int d^3x\int d^3y\,U(\mathbf{x}-\mathbf{y}) \;=\; V\int d^3x\,U(\mathbf{x}) \;=\; V\,\tilde{U}(\mathbf{0})$$

Where,

$$\tilde{U}(\mathbf{p}) \;=\; \int d^3x\,U(\mathbf{x})\,e^{i\mathbf{p}\cdot\mathbf{x}}$$

Thus, finally,

$$\frac{E^{(1)}}{N} \;=\; \frac{N-1}{2\,V}\,\tilde{U}(\mathbf{0})$$