Physics 443 Homework #2
Due Thursday, October 16, 2008

1.) Consider the path integral for a single point particle, with the action

$$S = \int_0^1 dt\big[p_\mu(t)\dot{x}^\mu(t) + \tfrac{1}{2}N(t)(p^2(t) - m^2 + i\epsilon)\big] \ .$$

This represents the quantization of the coordinates and momenta of the particle, subject to the mass shell constraint $p^2 = m^2$ (together with the $i\epsilon$ prescription) imposed by the Lagrange multiplier N. This action admits the reparametrization symmetry $\delta x = \alpha p$, $\delta p = 0$, $\delta N = -\partial_t \alpha$. This symmetry allows us to fix the gauge condition $N(t) = T$; the constant T must still be integrated over, however.

a) Path integrate over $x(t)$, subject to the boundary conditions $x^\mu(0) = x^\mu$, $x^\mu(1) = y^\mu$, yielding a delta function $\delta(\dot{p})$ along the path. Solve this constraint and path integrate over $p(t)$ to find the quantum mechanical propagation amplitude

$$\langle y|x\rangle = D_F(x-y) = \int_0^\infty dT(2\pi i T)^{-d/2}\exp\left[\frac{-i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right] \ ,$$

where d is the number of spacetime dimensions.

b) Use this integral representation to show that D_F satisfies

$$(\partial^2 + m^2)D_F = i\delta^{(4)}(x-y) \ .$$

c) Evaluate the T integral in terms of Bessel functions.

2.) Peskin and Schroeder 9.2a-c

Hints: For 9.2a, it is sufficient to *formulate* the partition function in terms of a path integral; you are going to evaluate it in part (b). For 9.2c, first show that the partition function can be formulated as a path integral over fields in Euclidean 4-space that are periodic in the imaginary time direction. The spatial field modes are harmonic oscillators; take the log of the partition function to get the free energy as a sum over modes of the free energy of each oscillator, and use your result for (b) to evaluate it.

A second approach to (c) uses the methods of problem (1). Use the representation

$$\begin{aligned}\log[Z] &= \log[\det(-\partial_E^2 + m^2)] = \mathrm{tr}\log(-\partial_E^2 + m^2)\\ &= \int d^4x \int_0^\infty \frac{dT}{T}\langle x|e^{-T(-\partial_E^2+m^2)}|x\rangle \ ,\end{aligned}$$

together with the representation of the matrix element derived in problem (1), to evaluate the functional determinant and hence the partition function. You may want to take $\partial/\partial m$ of the above expression to remove an m-independent divergence and render the integral finite.

3.) Write a field theory action describing *nonrelativistic* scalar particles interacting via a potential $U(\vec{x} - \vec{y})$ (this action-at-a-distance form of interaction is permissible in a nonrelativistic setting, but not in relativistic field theory, where it would break Lorentz invariance by selecting a preferred surface of simultaneity). Find the corresponding Hamiltonian for the field. Use your expression for the energy in terms of fields and evaluate it to show that the expectation value of the Hamiltonian in the noninteracting ground state of a system of N particles in a volume V is, to first order in perturbation theory,

$$\frac{E^{(1)}}{N} = \frac{N-1}{2V}\tilde{U}(0) \ ,$$

where

$$\tilde{U}(\vec{q}) = \int d^3x U(\vec{x})\exp[-i\vec{q}\cdot\vec{x}] \ .$$

Use of 'first quantized' methods to derive this answer is not acceptable (the point of the exercise is to gain familiarity with quantized fields; you may find it useful, however, to compare the two approaches).