QFT 1 : Problem Set 3

1.)

(a)

We consider the Dirac Lagrangian:

$$\mathcal{L}[\psi] = \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi$$

Where we have:

$$\{\,\gamma^\mu\,,\,\gamma^\nu\,\} = 2\,\eta^{\mu\nu} \qquad \text{and} \qquad \bar{\psi} = \psi^\dagger\gamma^0$$

And we use the conventions:

$$\partial\!\!\!/\psi \equiv \gamma^\mu\partial_\mu\psi \qquad \text{and} \qquad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

Lorentz invariance means that:

$$\mathcal{L}[\psi](x) = \mathcal{L}[\psi'](\Lambda x) \qquad \text{where} \qquad \psi'(\Lambda x) = M_\Lambda\psi(x)$$

Where $\Lambda \in SO(3,1)$ such that:

$$\Lambda^\mu{}_\nu = \left[\exp\left(-\tfrac{1}{2}\,i\,\omega_{\alpha\beta}\,L^{\alpha\beta}\right)\right]^\mu{}_\nu$$

Where,

$$\left[\,L^{\alpha\beta}\,\right]^\mu{}_\nu = i\left(\eta^{\alpha\mu}\delta^\beta{}_\nu - \eta^{\beta\mu}\delta^\alpha{}_\nu\right)$$

And where $M_\Lambda \in Spin(3,1)$ such that:

$$M_\Lambda = \exp\left(-\tfrac{1}{2}\,i\,\omega_{\alpha\beta}\,S^{\alpha\beta}\right)$$

Where,

$$S^{\alpha\beta} = \frac{i}{4}\left[\,\gamma^\alpha\,,\,\gamma^\beta\,\right]$$

Here we have suppressed spinor indices. We now consider

$$\mathcal{L}[\psi'](\Lambda x) = \bar{\psi}'(\Lambda x)\,(i\partial\!\!\!/\psi')(\Lambda x) - m\,\bar{\psi}'(\Lambda x)\psi'(\Lambda x)$$

Now,

$$M_\Lambda^\dagger\gamma^0 = \gamma^0 M_\Lambda^{-1} \qquad \text{since} \qquad \gamma^0\left(S^{\alpha\beta}\right)^\dagger\gamma^0 = -S^{\alpha\beta}$$

Thus,

$$\bar{\psi}'(\Lambda x) = \psi'^\dagger(\Lambda x)\gamma^0 = \psi^\dagger(x)M_\Lambda^\dagger\gamma^0 = \bar{\psi}(x)M_\Lambda^{-1}$$

Also,

$$(i\partial\!\!\!/\psi')(\Lambda x) = i\,\gamma^\mu\,\frac{\partial}{\partial y^\mu}\,\psi'(y)\bigg|_{y=\Lambda x} = i\,\gamma^\mu\,M_\Lambda\,\frac{\partial}{\partial y^\mu}\,\psi(\Lambda^{-1}y)\bigg|_{y=\Lambda x}$$

Defining $a = \Lambda^{-1}y$ and using the chain rule,

$$\frac{\partial}{\partial y^\mu}\,\psi(\Lambda^{-1}y)\bigg|_{y=\Lambda x} = \frac{\partial\psi}{\partial a^\nu}(a)\,\frac{\partial a^\nu}{\partial y^\mu}\bigg|_{y=\Lambda x} = \frac{\partial\psi}{\partial x^\nu}(x)\left(\Lambda^{-1}\right)^\nu{}_\mu$$

Thus,

$$(i\partial\!\!\!/\psi')(\Lambda x) = i\left(\Lambda^{-1}\right)^\nu{}_\mu\,\gamma^\mu\,M_\Lambda\,\partial_\nu\psi(x)$$

Thus we have:

$$\mathcal{L}\,[\psi']\,(\Lambda x) \;=\; \left(\Lambda^{-1}\right)^{\nu}{}_{\mu}\,\bar{\psi}(x)\,M_\Lambda^{-1}\gamma^\mu\,M_\Lambda\,i\partial_\nu\psi(x) \;-\; m\,\bar{\psi}(x)\psi(x)$$

Now, using the γ matrix algebra, for infinitesimal $\omega_{\alpha\beta}$, one may show:

$$\left(1 + i/2\,\omega_{\alpha\beta}\,S^{\alpha\beta}\right)\gamma^\mu\left(1 - i/2\,\omega_{\alpha\beta}\,S^{\alpha\beta}\right) \;=\; \left(1 - i/2\,\omega_{\alpha\beta}\,L^{\alpha\beta}\right)^{\mu}{}_{\nu}\gamma^\nu$$

Thus,

$$M_\Lambda^{-1}\gamma^\mu\,M_\Lambda \;=\; \Lambda^\mu{}_\nu\gamma^\nu$$

And, finally,

$$\mathcal{L}\,[\psi']\,(\Lambda x) \;=\; \mathcal{L}\,[\psi]\,(x) \;=\; \bar{\psi}(x)\,(i\partial\!\!\!/ - m)\,\psi(x)$$

(b)

We consider the Lagrangian of a massless Dirac field:

$$\mathcal{L}_0\,[\psi] \;=\; \bar{\psi}\,(i\partial\!\!\!/)\,\psi$$

A chiral rotation is given by:

$$\psi' \;=\; e^{\,i\alpha\gamma^5}\,\psi \qquad \text{where} \qquad \gamma^5 \;=\; i\gamma^0\gamma^1\gamma^2\gamma^3$$

This transformation can be seen to be a symmetry of the Lagrangian and thus of the action:

$$\mathcal{L}_0\,[\psi'] \;=\; \bar{\psi}'\,(i\partial\!\!\!/)\,\psi' \;=\; \psi^\dagger\,e^{\,-i\alpha(\gamma^5)^\dagger}\,\gamma^0\;(i\partial\!\!\!/)\;e^{\,i\alpha\gamma^5}\,\psi$$

Now, it may be shown,

$$(\gamma^5)^\dagger \;=\; \gamma^5 \qquad \text{and} \qquad \left\{\,\gamma^\mu\,,\,\gamma^5\,\right\} \;=\; 0$$

Thus,

$$\mathcal{L}_0\,[\psi'] \;=\; \psi^\dagger\,e^{\,-i\alpha\gamma^5}\,\gamma^0\,e^{\,-i\alpha\gamma^5}\;(i\partial\!\!\!/)\;\psi \;=\; \bar{\psi}\,(i\partial\!\!\!/)\,\psi \;=\; \mathcal{L}_0\,[\psi]$$

We now derive the Noether current. For an infinitesimal variation $\delta\psi$, after integrating by parts and applying the equations of motion, we find:

$$S\,[\psi + \delta\psi] - S\,[\psi] \;\simeq\; \int d^4x\;\partial_\mu\left(\,i\bar{\psi}\gamma^\mu\delta\psi\,\right)$$

Using $\delta\psi \;=\; i\epsilon\gamma^5\psi$ we find the conserved axial current:

$$J_A^\mu \;=\; \bar{\psi}\gamma^\mu\gamma^5\psi \qquad \text{where} \qquad \partial_\mu J_A^\mu \;=\; 0$$

This leads to the Noether charge:

$$Q_A \;=\; \int d^3x\,J_A^0 \;=\; \int d^3x\,\psi^\dagger\gamma^0\gamma^0\gamma^5\psi \;=\; \int d^3x\,\psi^\dagger\gamma^5\psi$$

Using,

$$\psi \;=\; \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad \text{and} \qquad \gamma^5 \;=\; \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

We find,

$$Q_A \;=\; \int d^3x\,\left(\,{\psi_R}^\dagger\psi_R \;-\; {\psi_L}^\dagger\psi_L\,\right)$$

To show that this operator measures the helicity of a collection of massless particles we introduce the matrices:

$$\sigma^\mu \equiv (\mathbf{1}, \boldsymbol{\sigma}) \qquad \text{and} \qquad \bar{\sigma}^\mu \equiv (\mathbf{1}, -\boldsymbol{\sigma})$$

The γ matrices take the form:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

This allows us to decouple the Dirac equation for massless particles:

$$i\bar{\sigma} \cdot \partial \psi_L = 0 \qquad \text{and} \qquad i\sigma \cdot \partial \psi_R = 0$$

We write the field operators as:

$$\psi_L = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_L(\mathbf{p})\, \alpha_L(\mathbf{p})\, e^{-ip\cdot x} + b^\dagger{}_L(\mathbf{p})\, \beta_L(\mathbf{p})\, e^{ip\cdot x} \right)$$

$$\psi_R = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_R(\mathbf{p})\, \alpha_R(\mathbf{p})\, e^{-ip\cdot x} + b^\dagger{}_R(\mathbf{p})\, \beta_R(\mathbf{p})\, e^{ip\cdot x} \right)$$

Imposing the equations of motion we find:

$$p \cdot \bar{\sigma} \alpha_L(\mathbf{p}) = p \cdot \bar{\sigma} \beta_L(\mathbf{p}) = 0 \qquad \text{where} \qquad p \cdot \bar{\sigma} = E_{\mathbf{p}} + \mathbf{p} \cdot \boldsymbol{\sigma}$$

And,

$$p \cdot \sigma \alpha_R(\mathbf{p}) = p \cdot \sigma \beta_R(\mathbf{p}) = 0 \qquad \text{where} \qquad p \cdot \sigma = E_{\mathbf{p}} - \mathbf{p} \cdot \boldsymbol{\sigma}$$

Thus, defining the helicity operator,

$$h = \mathbf{p} \cdot \boldsymbol{\sigma} / E_{\mathbf{p}}$$

We find:

$$h\, \alpha_L(\mathbf{p}) = -\alpha_L(\mathbf{p}) \qquad \text{and} \qquad h\, \beta_L(\mathbf{p}) = -\beta_L(\mathbf{p})$$

And,

$$h\, \alpha_R(\mathbf{p}) = \alpha_R(\mathbf{p}) \qquad \text{and} \qquad h\, \beta_R(\mathbf{p}) = \beta_R(\mathbf{p})$$

Using the orthogonality relations (see P+S pg 48):

$$\alpha_{L,R}{}^\dagger(\mathbf{p}) \alpha_{L,R}(\mathbf{p}) = \beta_{L,R}{}^\dagger(\mathbf{p}) \beta_{L,R}(\mathbf{p}) = 2E_{\mathbf{p}}$$

$$\alpha_{L,R}{}^\dagger(\mathbf{p}) \beta_{L,R}(-\mathbf{p}) = \beta_{L,R}{}^\dagger(-\mathbf{p}) \alpha_{L,R}(\mathbf{p}) = 0$$

We find:

$$:Q_A: = \int \frac{d^3p}{(2\pi)^3} \left(a^\dagger{}_R\, a_R + b^\dagger{}_R\, b_R - a^\dagger{}_L\, a_L - b^\dagger{}_L\, b_L \right)$$

Thus we see that the Noether charge measures the helicity of a collection of particles.

We now add a mass term to the Lagrangian and verify that the chiral symmetry is violated.

$$\mathcal{L}[\psi] = \bar{\psi}\, (i\partial\!\!\!/ - m)\, \psi = \mathcal{L}_0[\psi] - m\bar{\psi}\psi$$

Thus,

$$\mathcal{L}[\psi'] = \mathcal{L}_0[\psi'] - m\, \psi^\dagger e^{-i\alpha\gamma^5}\, \gamma^0\, e^{i\alpha\gamma^5}\, \psi = \mathcal{L}_0[\psi] - m\, \bar{\psi}\, e^{2i\alpha\gamma^5}\, \psi$$

Thus for an infinitesimal variation $\delta\psi = i\epsilon\gamma^5\psi$ we find:

$$S[\psi + \delta\psi] - S[\psi] \simeq -\epsilon \int d^4x \left(2im\, \bar{\psi}\gamma^5\psi \right)$$

Thus the chiral symmetry is violated by the mass term. From above:

$$S\,[\psi + \delta\psi] - S\,[\psi] \;\simeq\; -\,\epsilon \int d^4x\; \partial_\mu \left(\bar{\psi}\gamma^\mu\gamma^5\psi \right)$$

Thus we may express the violation of current conservation as:

$$\partial_\mu \left(\bar{\psi}\gamma^\mu\gamma^5\psi \right) \;=\; 2im\,\bar{\psi}\gamma^5\psi$$

It is simpler to derive this by computing $\partial_\mu J^\mu_A$ and making use of the Dirac equation.

(c)

We now consider the symmetry under phase rotations:

$$\psi' \;=\; e^{\,i\alpha}\,\psi$$

Clearly,

$$\mathcal{L}\,[\psi'] \;=\; \mathcal{L}\,[\psi]$$

For an infinitesimal variation $\delta\psi \;=\; i\epsilon\psi$ we find:

$$S\,[\psi + \delta\psi] - S\,[\psi] \;\simeq\; -\,\epsilon \int d^4x\; \partial_\mu \left(\bar{\psi}\gamma^\mu\psi \right)$$

Thus we find the conserved current:

$$J^\mu \;=\; \bar{\psi}\gamma^\mu\psi \qquad \text{where} \qquad \partial_\mu J^\mu \;=\; 0$$

2.) Peskin & Schroeder 3.1 : Lorentz group

We begin with the Lorentz algebra:

$$[\,J^{\mu\nu}\,,\,J^{\rho\sigma}\,] \;=\; i\,(\,\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}\,)$$

(a)

We define the generators of rotations and boosts as:

$$L^i \;=\; \tfrac{1}{2}\,\epsilon^{ijk}\,J^{jk} \qquad \text{and} \qquad K^i \;=\; J^{0i}$$

We compute:

$$\left[\,L^i\,,\,L^j\,\right] \;=\; \tfrac{1}{4}\,\epsilon^{ipq}\,\epsilon^{jmn}\,[\,J^{pq}\,,\,J^{mn}\,] \;=\; i\,\epsilon^{ipq}\,\epsilon^{jmn}\,\delta^{pm}\,J^{qn}$$

Where we have substituted $\delta^{ij} = -\,\eta^{ij}$. Now,

$$J^{ij} \;=\; \epsilon^{ijk}\,L^k \qquad \text{and} \qquad \epsilon^{ipq}\,\epsilon^{jmn}\,\delta^{pm} \;=\; \left(\,\delta^{ij}\delta^{qn} - \delta^{in}\delta^{qj}\,\right)$$

Thus,

$$\left[\,L^i\,,\,L^j\,\right] \;=\; i\,\left(\,\delta^{ij}\delta^{qn} - \delta^{in}\delta^{qj}\,\right)\,\epsilon^{qnk}\,L^k \;=\; i\,\epsilon^{ijk}\,L^k$$

We next compute:

$$\left[\,K^i\,,\,K^j\,\right] \;=\; \left[\,J^{0i}\,,\,J^{0j}\,\right] \;=\; -i\,\eta^{00}\,J^{ij} \;=\; -i\,\epsilon^{ijk}\,L^k$$

And, finally we compute:

$$\left[\, L^i\,,\, K^j\,\right] \;=\; \tfrac{1}{2}\,\epsilon^{ipq}\,\left[\, J^{pq}\,,\, J^{0j}\,\right] \;=\; -i\,\epsilon^{ipq}\,\delta^{pj}\,J^{q0} \;=\; i\,\epsilon^{ijk}\,K^k$$

We now define:

$$\mathbf{J}_{\pm} \;=\; \tfrac{1}{2}\,(\,\mathbf{L} \pm i\,\mathbf{K}\,)$$

We compute:

$$\begin{aligned}\left[\, J_+^i\,,\, J_+^j\,\right] \;&=\; \tfrac{1}{4}\,\big(\,[\,J^i\,,\,J^j\,] - [\,K^i\,,\,K^j\,] + i\,[\,J^i\,,\,K^j\,] + i\,[\,K^i\,,\,J^j\,]\,\big) \\ &=\; i\,\epsilon^{ijk}\,\tfrac{1}{2}\,\big(\,L^k \,+\, i\,K^k\,\big) \;=\; i\,\epsilon^{ijk}\,J_+^k\end{aligned}$$

Similarly,

$$\begin{aligned}\left[\, J_-^i\,,\, J_-^j\,\right] \;&=\; \tfrac{1}{4}\,\big(\,[\,J^i\,,\,J^j\,] - [\,K^i\,,\,K^j\,] - i\,[\,J^i\,,\,K^j\,] - i\,[\,K^i\,,\,J^j\,]\,\big) \\ &=\; i\,\epsilon^{ijk}\,\tfrac{1}{2}\,\big(\,L^k \,-\, i\,K^k\,\big) \;=\; i\,\epsilon^{ijk}\,J_-^k\end{aligned}$$

And,

$$\left[\, J_+^i\,,\, J_-^j\,\right] \;=\; \tfrac{1}{4}\,\big(\,[\,J^i\,,\,J^j\,] + [\,K^i\,,\,K^j\,] - i\,[\,J^i\,,\,K^j\,] + i\,[\,K^i\,,\,J^j\,]\,\big) \;=\; 0$$

Thus we find that we may put the representations of the Lorentz group in correspondence with the representations of $SU(2) \times SU(2)$. Note that this does not mean that $Spin(3,1) = SL(2,\mathbb{C})$ is the same Lie group as $SU(2) \times SU(2)$ since $SO(3,1)$ and thus $Spin(3,1)$ are non-compact while $SU(2) \times SU(2)$ is compact. $SU(2) \times SU(2)$ is the same manifold as the product $S^3 \times S^3$. $Spin(3,1)$ is topologically $S^3 \times \mathbb{R}^3$.

(b)

We consider the (j_+, j_-) representation of $SU(2)_+ \times SU(2)_-$. A field in this representation transforms under an infinitesimal $\Lambda \in SO(3,1)$ as:

$$\tilde{\Phi}_{(j_+,j_-)}(\Lambda x) \;=\; (\,\mathbf{1} - i\boldsymbol{\theta}\cdot\mathbf{L} - i\boldsymbol{\beta}\cdot\mathbf{K}\,)\,\Phi_{(j_+,j_-)}(x)$$

For the $(1/2, 0)$ representation we have:

$$\mathbf{J}_+ \;=\; \boldsymbol{\sigma}/2 \;=\; \tfrac{1}{2}\,(\,\mathbf{L} \,+\, i\,\mathbf{K}\,) \qquad \text{and} \qquad \mathbf{J}_- \;=\; 0 \;=\; \tfrac{1}{2}\,(\,\mathbf{L} \,-\, i\,\mathbf{K}\,)$$

Thus $\mathbf{L} = i\mathbf{K} = \boldsymbol{\sigma}/2$ and,

$$\tilde{\Phi}_{(1/2,0)}(\Lambda x) \;=\; (\,\mathbf{1} - i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2 - \boldsymbol{\beta}\cdot\boldsymbol{\sigma}/2\,)\,\Phi_{(1/2,0)}(x)$$

For the $(0, 1/2)$ representation we have:

$$\mathbf{J}_+ \;=\; 0 \;=\; \tfrac{1}{2}\,(\,\mathbf{L} \,+\, i\,\mathbf{K}\,) \qquad \text{and} \qquad \mathbf{J}_- \;=\; \boldsymbol{\sigma}/2 \;=\; \tfrac{1}{2}\,(\,\mathbf{L} \,-\, i\,\mathbf{K}\,)$$

Thus $\mathbf{L} = -i\mathbf{K} = \boldsymbol{\sigma}/2$ and,

$$\tilde{\Phi}_{(0,1/2)}(\Lambda x) \;=\; (\,\mathbf{1} - i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2 + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}/2\,)\,\Phi_{(0,1/2)}(x)$$

Thus in the notation of P&S we have:

$$\Phi_{(1/2,0)} \;\equiv\; \psi_L \qquad \text{and} \qquad \Phi_{(0,1/2)} \;\equiv\; \psi_R$$

(c)

We consider the behavior of the matrix,

$$\mathbb{V} \;=\; \begin{pmatrix} V^0+V^3 & V^1-iV^2 \\ V^1-iV^2 & V^0-V^3 \end{pmatrix} \;=\; V_\mu\,\bar{\sigma}^\mu \qquad\text{where}\qquad V^\mu \;=\; \tfrac{1}{2}\,\mathrm{tr}\,(\,\sigma^\mu\,\mathbb{V}\,)$$

under the infinitesimal transformation:

$$\tilde{\mathbb{V}} \;=\; \left(\,\mathbf{1}+\tfrac{1}{2}(\boldsymbol{\beta}-i\boldsymbol{\theta})\cdot\boldsymbol{\sigma}\,\right)\mathbb{V}\left(\,\mathbf{1}+\tfrac{1}{2}(\boldsymbol{\beta}+i\boldsymbol{\theta})\cdot\boldsymbol{\sigma}\,\right)$$

Or using the cyclicity of the trace and defining $\mathbf{a}=(\boldsymbol{\beta}+i\boldsymbol{\theta})/2$:

$$\tilde{V}^\mu \;=\; \tfrac{1}{2}\,\mathrm{tr}\left[\,(\,\mathbf{1}+\mathbf{a}\cdot\boldsymbol{\sigma}\,)\;\sigma^\mu\;(\,\mathbf{1}+\bar{\mathbf{a}}\cdot\boldsymbol{\sigma}\,)\;\left(\,V^0+\mathbf{V}\cdot\boldsymbol{\sigma}\,\right)\,\right]$$

Now, to first order in $\mathbf{a}$:

$$(\,\mathbf{1}+\mathbf{a}\cdot\boldsymbol{\sigma}\,)\;\sigma^0\;(\,\mathbf{1}+\bar{\mathbf{a}}\cdot\boldsymbol{\sigma}\,) \;=\; (\,\mathbf{1}+\boldsymbol{\beta}\cdot\boldsymbol{\sigma}\,)$$

And, using $\sigma^i\sigma^j=\delta^{ij}+i\epsilon^{ijk}\sigma^k$:

$$(\,\mathbf{1}+\mathbf{a}\cdot\boldsymbol{\sigma}\,)\;\boldsymbol{\sigma}\;(\,\mathbf{1}+\bar{\mathbf{a}}\cdot\boldsymbol{\sigma}\,) \;=\; \boldsymbol{\sigma}\;+\;\boldsymbol{\beta}\;+\;\boldsymbol{\theta}\times\boldsymbol{\sigma}$$

Thus,

$$\tilde{V}^0 \;=\; V^0\;+\;\boldsymbol{\beta}\cdot\mathbf{V} \qquad\text{and}\qquad \tilde{\mathbf{V}} \;=\; \mathbf{V}\;+\;\boldsymbol{\beta}\,V^0\;+\;\boldsymbol{\theta}\times\mathbf{V}$$

This is just the transformation of a Lorentz vector.

3.) Peskin & Schroeder 3.4 : Majorana fermions

(a)

Borrowing notation from problems 1 and 2 above, we define

$$M_\Lambda \;=\; \exp\left(-\tfrac{1}{2}\,i\,\omega_{\alpha\beta}\,S^{\alpha\beta}\,\right) \;=\; \begin{pmatrix} \Omega_L & 0 \\ 0 & \Omega_R \end{pmatrix}$$

Where,

$$\Omega_L \;=\; \exp\left(-\tfrac{1}{2}\,i\,(\boldsymbol{\theta}-i\boldsymbol{\beta})\cdot\boldsymbol{\sigma}\,\right)$$

And using $\sigma^2\boldsymbol{\sigma}\sigma^2=-\boldsymbol{\sigma}^*$ we have

$$\Omega_R^* \;=\; \sigma^2\,\Omega_L\,\sigma^2$$

And,

$$\bar{\sigma}^\mu \;=\; \sigma^2\,(\sigma^\mu)^*\,\sigma^2$$

From the properties of the γ matrices we have

$$\Omega_R^{-1}\bar{\sigma}^\mu\Omega_L \;=\; \Lambda^\mu{}_\nu\,\bar{\sigma}^\nu \qquad\qquad \Omega_L^{-1}\sigma^\mu\Omega_R \;=\; \Lambda^\mu{}_\nu\,\sigma^\nu$$

And,

$$\sigma^\mu\bar{\sigma}^\nu+\sigma^\nu\bar{\sigma}^\mu \;=\; \bar{\sigma}^\mu\sigma^\nu+\bar{\sigma}^\nu\sigma^\mu \;=\; 2\,\eta^{\mu\nu}$$

We consider the equation of motion for left-handed Majorana fermions

$$i\bar{\sigma}\cdot\partial\chi-im\sigma^2\,\chi^* \;=\; 0$$

Where χ transforms as

$$\chi'(\Lambda x) \,=\, \Omega_L \chi(x)$$

Now,

$$-im\sigma^2 (\chi')^*(\Lambda x) \,=\, -im\sigma^2 \Omega_L^* \chi^*(x) \,=\, -im\Omega_R \sigma^2 \chi^*(x)$$

As in problem 1 above,

$$i\bar{\sigma}\cdot\partial\chi'(\Lambda x) \,=\, i\big(\Lambda^{-1}\big)^{\nu}{}_{\mu}\, \bar{\sigma}^\mu\, \Omega_L\, \partial_\nu \chi(x)$$

Or,

$$i\bar{\sigma}\cdot\partial\chi'(\Lambda x) \,=\, i\,\Omega_R\, \bar{\sigma}\cdot\partial\chi(x)$$

Thus the equations of motion are Lorentz invariant

$$i\bar{\sigma}\cdot\partial\chi' - im\sigma^2\,(\chi')^* \,=\, 0$$

Complex conjugating the equations of motion

$$-i(\bar{\sigma}^\mu)^*\partial_\mu\chi^* - im\sigma^2\,\chi \,=\, -i\sigma^2\sigma^\mu\sigma^2\partial_\mu\chi^* - im\sigma^2\,\chi \,=\, 0$$

Operating on this equation with $i(\bar{\sigma}\cdot\partial)\sigma^2$ we find

$$\bar{\sigma}^\mu\sigma^\nu\sigma^2\partial_\nu\partial_\mu\chi^* + m^2\sigma^2\chi^* \,=\, 0$$

Using,

$$\bar{\sigma}^\mu\sigma^\nu\partial_\mu\partial_\nu \,=\, \tfrac{1}{2}\left(\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\mu\sigma^\nu\right)\partial_\mu\partial_\nu \,=\, \partial^2$$

We find that χ satisfies the Klein-Gordon equation

$$\left(\partial^2 + m^2\right)\chi \,=\, 0$$

(b)

The action for Majorana fermions is

$$S[\chi,\chi^*] \,=\, \int d^4x\left[\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \tfrac{1}{2}im\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right]$$

To show that $S = S^*$ we first compute

$$\left[\tfrac{1}{2}im\left(\chi_a(\sigma^2)_{ab}\chi_b - \chi_a^*(\sigma^2)_{ab}\chi_b^*\right)\right]^* \,=\, \left[\tfrac{1}{2}im\left(\chi_b^*(\sigma^2)_{ab}\chi_a^* - \chi_b(\sigma^2)_{ab}\chi_a\right)\right]$$

Where we have used $(\chi_a\chi_b)^* = \chi_b^*\chi_a^*$ and $(\sigma^2)^* = -\sigma^2$. Using $\chi_a\chi_b = -\chi_b\chi_a$ we find

$$\left[\tfrac{1}{2}im\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right]^* \,=\, \left[\tfrac{1}{2}im\left(\chi_a(\sigma^2)_{ab}\chi_b - \chi_a^*(\sigma^2)_{ab}\chi_b^*\right)\right]$$

Using $(\bar{\sigma}^\mu)^\dagger$ we consider

$$\left[i\,\chi_a^*\bar{\sigma}^\mu_{ab}\partial_\mu\chi_b\right]^* \,=\, \left[-i\,\bar{\sigma}^\mu_{ba}\partial_\mu\chi_b^*\chi_a\right]$$

Ignoring surface terms we find

$$\left[\chi^\dagger i\bar{\sigma}\cdot\partial\chi\right]^* \,=\, \left[i\,\bar{\sigma}^\mu_{ba}\chi_b^*\partial_\mu\chi_a\right]$$

Thus $S = S^*$. Varying S with respect to χ^*

$$S[\chi,\chi^* + (\delta\chi)^*] - S[\chi,\chi^*] \,\simeq\, \int d^4x\left[(\delta\chi)^\dagger i\bar{\sigma}\cdot\partial\chi - im\,(\delta\chi)^\dagger\sigma^2\chi^*\right]$$

This leads directly to the equations of motion

$$i\bar{\sigma}\cdot\partial\chi - im\sigma^2\,\chi^* \,=\, 0$$

(c)

We write the Dirac Lagrangian as

$$\mathcal{L} \;=\; \left(\begin{array}{cc} \psi_L^\dagger & \psi_R^\dagger \end{array}\right) \left(\begin{array}{cc} i\bar\sigma\cdot\partial & -m \\ -m & i\sigma\cdot\partial \end{array}\right) \left(\begin{array}{c} \psi_L \\ \psi_R \end{array}\right)$$

Setting $\psi_L = \chi_1$ and $\psi_R = i\sigma^2\chi_2^*$

$$\mathcal{L} \;=\; \chi_1^\dagger i\bar\sigma\cdot\partial\chi_1 + \chi_2^T i(\sigma^2\sigma\sigma^2)\cdot\partial\chi_2^* - im\chi_1^\dagger\sigma^2\chi_2^* + im\chi_2^T\sigma^2\chi_1$$

Note that since (χ_1,χ_2) are Grassmann variables

$$\chi_2^T i(\sigma^2\sigma\sigma^2)\cdot\partial\chi_2^* \;=\; -\partial\chi_2^\dagger\cdot i\bar\sigma\chi_2$$

Thus, integrating by parts

$$\mathcal{L} \;=\; \chi_1^\dagger i\bar\sigma\cdot\partial\chi_1 + \chi_2^\dagger i\bar\sigma\cdot\partial\chi_2 - im\chi_1^\dagger\sigma^2\chi_2^* + im\chi_2^T\sigma^2\chi_1$$

If we set $\chi\equiv\chi_1=\chi_2$ we find (twice) the Majorana Lagrangian. If we make this identification then the $U(1)$ symmetry $\chi_1\to e^{i\theta}\chi_1$ and $\chi_2\to e^{-i\theta}\chi_2$ of the Dirac Lagrangian is lost.

(e)

Consider the following form of the Weyl spinor components of a solution to the Dirac equation

$$\psi_L(x) \;=\; \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\sqrt{p\cdot\sigma}\left(\alpha_s(\mathbf{p})\,\xi_s\,e^{-ip\cdot x} + \beta_s^*(\mathbf{p})\,(-i\sigma^2\xi_s^*)\,e^{ip\cdot x}\right)$$

$$\psi_R(x) \;=\; \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\sqrt{p\cdot\bar\sigma}\left(\alpha_s(\mathbf{p})\,\xi_s\,e^{-ip\cdot x} - \beta_s^*(\mathbf{p})\,(-i\sigma^2\xi_s^*)\,e^{ip\cdot x}\right)$$

Here we have used the conventions of Peskin and Schroeder. The Majorana fermions satisfy the condition

$$\chi \;=\; \psi_L(x) \;=\; -i\sigma^2\,\psi_R^*(x)$$

Imposing this condition, after some algebra we find

$$\alpha_s(\mathbf{p}) \;=\; \beta_s(\mathbf{p})$$

Promoting $\alpha_s(\mathbf{p})$ to an operator $\hat a_s(\mathbf{p})$, χ may be written as

$$\chi(x) \;=\; \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\sqrt{p\cdot\sigma}\left(\hat a_s(\mathbf{p})\,\xi_s\,e^{-ip\cdot x} + \hat a_s^\dagger(\mathbf{p})\,(-i\sigma^2\xi_s^*)\,e^{ip\cdot x}\right)$$

It may be verified that this satisfies the Majorana equation. Furthermore it may be seen that the canonical anti-commutation relations

$$\{\chi_a(\mathbf{x}),\chi_b^\dagger(\mathbf{y})\} \;=\; \delta_{ab}\,\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

follow from the anti-commutation relations

$$\{\hat a_s(\mathbf{p}),\hat a_r^\dagger(\mathbf{k})\} \;=\; (2\pi)^3\,\delta_{sr}\,\delta^{(3)}(\mathbf{p}-\mathbf{k})$$

To see this we write

$$\chi(x) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\sum_s \left(\hat{a}_s(\mathbf{p})\,A_s(p)\,e^{-ip\cdot x} + \hat{a}_s^\dagger(\mathbf{p})\,B_s(p)\,e^{ip\cdot x}\right)$$

Where

$$A_s(p) \;=\; \sqrt{p\cdot\sigma}\,\xi_s \qquad\qquad B_s(p) \;=\; \sqrt{p\cdot\sigma}\,(-i\sigma^2)\,\xi_s^*$$

Setting $x^0 = y^0 = 0$ and imposing the anti-commuation relations

$$\{\chi_a(\mathbf{x}),\chi_b^\dagger(\mathbf{y})\} \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,\sum_s \Big((A_s(p))_a\,(A_s(p))_b^* e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + (B_s(p))_a\,(B_s(p))_b^* e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Big)$$

It may be verified that

$$\sum_s (A_s(p))_a\,(A_s(p))_b^* \;=\; \sum_s (B_s(p))_a\,(B_s(p))_b^* \;=\; (\,p\cdot\sigma\,)_{ab}$$

This leads directly to the canonical anti-commutation relations. The Hamiltonian may be constructed from the Lagrangian for Majorana fermions. However it is straightforward to see that the Hamiltonian takes the form expected from the projection of the Dirac equation

$$H \;=\; \int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\sum_s \hat{a}_s^\dagger(\mathbf{p})\,\hat{a}_s(\mathbf{p})$$

That is we simply identify the $\hat{a}_s(\mathbf{p})$ and $\hat{b}_s(\mathbf{p})$ operators and, by convention, divide the action and thus the Hamiltonian by two.

4.) Peskin & Schroeder 3.7 : PCT

(a)

We consider the discrete transformation properties of:

$$\bar{\psi}\,\sigma^{\mu\nu}\,\psi \qquad\text{where}\qquad \sigma^{\mu\nu} \;=\; \frac{i}{2}\,[\,\gamma^\mu\,,\,\gamma^\nu\,]$$

We begin with the parity operator $\mathcal{P} = \mathcal{P}^\dagger = \mathcal{P}^{-1}$:

$$\mathcal{P}\,\psi(t,\mathbf{x})\,\mathcal{P} \;=\; \gamma^0\,\psi(t,-\mathbf{x})$$

Now, since $(\gamma^0)^\dagger = \gamma^0$,

$$\mathcal{P}\,\bar{\psi}(t,\mathbf{x})\,\mathcal{P} \;=\; (\mathcal{P}\,\psi(t,\mathbf{x})\,\mathcal{P})^\dagger\;\gamma^0 \;=\; \bar{\psi}(t,-\mathbf{x})\,\gamma^0$$

Thus,

$$\left(\mathcal{P}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{P}\right)(t,\mathbf{x}) \;=\; \left(\mathcal{P}\,\bar{\psi}\,\mathcal{P}\,\sigma^{\mu\nu}\,\mathcal{P}\,\psi\,\mathcal{P}\right)(t,\mathbf{x}) \;=\; \left(\bar{\psi}\,\gamma^0\,\sigma^{\mu\nu}\,\gamma^0\,\psi\right)(t,-\mathbf{x})$$

Now,

$$\gamma^0\,\sigma^{0j}\,\gamma^0 \;=\; -\,\sigma^{0j} \qquad\text{and}\qquad \gamma^0\,\sigma^{jk}\,\gamma^0 \;=\; \sigma^{jk}$$

Thus,

$$\left(\mathcal{P}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{P}\right)(t,\mathbf{x}) \;=\; \eta^{\mu\mu}\,\eta^{\nu\nu}\;\left(\bar{\psi}\,\sigma^{\mu\nu}\,\psi\right)(t,-\mathbf{x})$$

Where no summation is implied.

Next we consider the time reversal operator $\mathcal{T} = \mathcal{T}^\dagger = \mathcal{T}^{-1}$:

$$\mathcal{T}\,\psi(t,\mathbf{x})\,\mathcal{T} \;=\; -\,\gamma^1\,\gamma^3\,\psi(-t,\mathbf{x})$$

$\mathcal{T}$ is an anti-linear operator so that:

$$\mathcal{T}\,\bar{\psi}(t,\mathbf{x})\,\mathcal{T} \;=\; \left(\mathcal{T}\,\psi(t,\mathbf{x})\,\mathcal{T}\right)^\dagger\,(\gamma^0)^* \;=\; \left(-\,\gamma^1\,\gamma^3\,\psi(-t,\mathbf{x})\right)^\dagger\,\gamma^0$$

Since $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ we find:

$$\mathcal{T}\,\psi(t,\mathbf{x})\,\mathcal{T} \;=\; -\,\psi^\dagger(-t,\mathbf{x})\,(\gamma^1)^\dagger\,(\gamma^3)^\dagger\,\gamma^0 \;=\; \bar{\psi}(-t,\mathbf{x})\,\gamma^1\,\gamma^3$$

Thus,

$$\begin{aligned}\left(\mathcal{T}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{T}\right)(t,\mathbf{x}) \;&=\; \left(\mathcal{T}\,\bar{\psi}\,\mathcal{T}\,(\,\sigma^{\mu\nu}\,)^*\,\mathcal{T}\,\psi\,\mathcal{T}\right)(t,\mathbf{x})\\ &=\; \left(\bar{\psi}\,\gamma^1\,\gamma^3\,(\,\sigma^{\mu\nu}\,)^*\,\gamma^3\,\gamma^1\,\psi\right)(-t,\mathbf{x})\end{aligned}$$

Now, in the Weyl representation:

$$(\gamma^0)^* \;=\; \gamma^0 \qquad (\gamma^1)^* \;=\; \gamma^1 \qquad (\gamma^2)^* \;=\; -\,\gamma^2 \qquad (\gamma^3)^* \;=\; \gamma^3$$

And,

$$(\,\sigma^{\mu\nu}\,)^* \;=\; -\,\frac{i}{2}\,[\,(\gamma^\mu)^*\,,\,(\gamma^\nu)^*\,]$$

Thus, using the γ matrix algebra:

$$\begin{aligned}\gamma^1\,\gamma^3\,\left(\sigma^{12}\right)^*\gamma^3\,\gamma^1 \;&=\; \gamma^1\,\gamma^3\,\sigma^{12}\,\gamma^3\,\gamma^1 &&=\; -\,\sigma^{12}\\ \gamma^1\,\gamma^3\,\left(\sigma^{23}\right)^*\gamma^3\,\gamma^1 \;&=\; \gamma^1\,\gamma^3\,\sigma^{23}\,\gamma^3\,\gamma^1 &&=\; -\,\sigma^{23}\\ \gamma^1\,\gamma^3\,\left(\sigma^{31}\right)^*\gamma^3\,\gamma^1 \;&=\; -\,\gamma^1\,\gamma^3\,\sigma^{31}\,\gamma^3\,\gamma^1 &&=\; -\,\sigma^{31}\end{aligned}$$

And,

$$\begin{aligned}\gamma^1\,\gamma^3\,\left(\sigma^{01}\right)^*\gamma^3\,\gamma^1 \;&=\; -\,\gamma^1\,\gamma^3\,\sigma^{01}\,\gamma^3\,\gamma^1 &&=\; \sigma^{01}\\ \gamma^1\,\gamma^3\,\left(\sigma^{02}\right)^*\gamma^3\,\gamma^1 \;&=\; \gamma^1\,\gamma^3\,\sigma^{02}\,\gamma^3\,\gamma^1 &&=\; \sigma^{02}\\ \gamma^1\,\gamma^3\,\left(\sigma^{03}\right)^*\gamma^3\,\gamma^1 \;&=\; -\,\gamma^1\,\gamma^3\,\sigma^{03}\,\gamma^3\,\gamma^1 &&=\; \sigma^{03}\end{aligned}$$

Thus, again with no summation implied:

$$\left(\mathcal{T}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{T}\right)(t,\mathbf{x}) \;=\; -\,\eta^{\mu\mu}\,\eta^{\nu\nu}\,\left(\bar{\psi}\,\sigma^{\mu\nu}\,\psi\right)(-t,\mathbf{x})$$

We now consider the charge conjugation operator: $\mathcal{C} = \mathcal{C}^\dagger = \mathcal{C}^{-1}$:

$$\mathcal{C}\,\psi\,\mathcal{C} \;=\; -i\,\left(\bar{\psi}\,\gamma^0\,\gamma^2\right)^T$$

Thus,

$$\mathcal{C}\,\bar{\psi}\,\mathcal{C} \;=\; (\,\mathcal{C}\,\psi\,\mathcal{C}\,)^\dagger\,\gamma^0 \;=\; -i\,\left(\left(\bar{\psi}\,\gamma^0\,\gamma^2\right)^\dagger\right)^T\gamma^0 \;=\; i\,\left(\gamma^2\,\psi\right)^T\,\gamma^0$$

And,

$$\mathcal{C}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{C} \;=\; \mathcal{C}\,\bar{\psi}\,\mathcal{C}\,\sigma^{\mu\nu}\,\mathcal{C}\,\psi\,\mathcal{C} \;=\; \left(\gamma^2\,\psi\right)^T\,\gamma^0\,\sigma^{\mu\nu}\,\left(\bar{\psi}\,\gamma^0\,\gamma^2\right)^T$$

Now, in the Weyl representation:

$$(\gamma^0)^T \;=\; \gamma^0 \qquad (\gamma^1)^T \;=\; -\gamma^1 \qquad (\gamma^2)^T \;=\; \gamma^2 \qquad (\gamma^3)^T \;=\; -\gamma^3$$

Thus,

$$\mathcal{C}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{C} \;=\; \psi^T\,\gamma^2\,\gamma^0\,\sigma^{\mu\nu}\,\gamma^2\,\gamma^0\,\bar{\psi}^T \;=\; \bar{\psi}\,\gamma^0\,\gamma^2\,\left(\sigma^{\mu\nu}\right)^T\,\gamma^0\,\gamma^2\,\psi$$

Now,

$$\left(\sigma^{\mu\nu}\right)^T \;=\; -\,\frac{i}{2}\,\left[\,(\gamma^\mu)^T\,,\,(\gamma^\nu)^T\,\right]$$

Thus, using the γ matrix algebra:

$$\begin{aligned}
\gamma^0\,\gamma^2\left(\sigma^{12}\right)^T\gamma^0\,\gamma^2 \;&=\; \gamma^0\,\gamma^2\,\sigma^{12}\,\gamma^0\,\gamma^2 &&=\; -\,\sigma^{12} \\
\gamma^0\,\gamma^2\left(\sigma^{23}\right)^T\gamma^0\,\gamma^2 \;&=\; \gamma^0\,\gamma^2\,\sigma^{23}\,\gamma^0\,\gamma^2 &&=\; -\,\sigma^{23} \\
\gamma^0\,\gamma^2\left(\sigma^{31}\right)^T\gamma^0\,\gamma^2 \;&=\; -\,\gamma^0\,\gamma^2\,\sigma^{31}\,\gamma^0\,\gamma^2 &&=\; -\,\sigma^{31}
\end{aligned}$$

And,

$$\begin{aligned}
\gamma^0\,\gamma^2\left(\sigma^{01}\right)^T\gamma^0\,\gamma^2 \;&=\; \gamma^0\,\gamma^2\,\sigma^{01}\,\gamma^0\,\gamma^2 &&=\; -\,\sigma^{01} \\
\gamma^0\,\gamma^2\left(\sigma^{02}\right)^T\gamma^0\,\gamma^2 \;&=\; -\,\gamma^0\,\gamma^2\,\sigma^{02}\,\gamma^0\,\gamma^2 &&=\; -\,\sigma^{02} \\
\gamma^0\,\gamma^2\left(\sigma^{03}\right)^T\gamma^0\,\gamma^2 \;&=\; \gamma^0\,\gamma^2\,\sigma^{03}\,\gamma^0\,\gamma^2 &&=\; -\,\sigma^{03}
\end{aligned}$$

Thus,

$$\mathcal{C}\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi\,\mathcal{C} \;=\; -\,\bar{\psi}\,\sigma^{\mu\nu}\,\psi$$

(b)

We now consider the complex scalar field:

$$\phi(x) \;=\; \int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\left(a_{\mathbf{p}}\,e^{-ip\cdot x} \,+\, b^\dagger_{\mathbf{p}}\,e^{ip\cdot x}\right)$$

The parity operator acts as:

$$\mathcal{P}\,a_{\mathbf{p}}\,\mathcal{P} \;=\; a_{-\mathbf{p}} \qquad \text{and} \qquad \mathcal{P}\,b_{\mathbf{p}}\,\mathcal{P} \;=\; b_{-\mathbf{p}}$$

Thus,

$$\begin{aligned}
\mathcal{P}\,\phi(t,\mathbf{x})\,\mathcal{P} \;&=\; \int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\Big(a_{-\mathbf{p}}\,e^{-ip\cdot x} \,+\, b^\dagger_{-\mathbf{p}}\,e^{ip\cdot x}\Big) \\
&=\; \int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\left(a_{\mathbf{p}}\,e^{-iE_{\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}} \,+\, b^\dagger_{\mathbf{p}}\,e^{iE_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}}\right) \\
&=\; \phi(t,-\mathbf{x})
\end{aligned}$$

The anti-linear time reversal operator acts as:

$$\mathcal{T}\, a_{\mathbf{p}}\, \mathcal{T} \,=\, a_{-\mathbf{p}} \qquad \text{and} \qquad \mathcal{T}\, b_{\mathbf{p}}\, \mathcal{T} \,=\, b_{-\mathbf{p}}$$

Thus,

$$\begin{aligned}
\mathcal{T}\,\phi(t,\mathbf{x})\,\mathcal{T} \;&=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\left(a_{-\mathbf{p}}\,e^{ip\cdot x} \,+\, b^{\dagger}_{-\mathbf{p}}\,e^{-ip\cdot x}\right)\\
&=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\left(a_{\mathbf{p}}\,e^{iE_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}} \,+\, b^{\dagger}_{\mathbf{p}}\,e^{-iE_{\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}}\right)\\
&=\; \phi(-t,\mathbf{x})
\end{aligned}$$

The charge conjugation operator acts as:

$$\mathcal{C}\, a_{\mathbf{p}}\, \mathcal{C} \,=\, b_{\mathbf{p}} \qquad \text{and} \qquad \mathcal{C}\, b_{\mathbf{p}}\, \mathcal{C} \,=\, a_{\mathbf{p}}$$

Thus,

$$\mathcal{C}\,\phi(x)\,\mathcal{C} \,=\, \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\left(b_{\mathbf{p}}\,e^{-ip\cdot x} \,+\, a^{\dagger}_{\mathbf{p}}\,e^{ip\cdot x}\right) \,=\, \phi(x)^{\dagger}$$

We now consider the transformation properties under $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{T}$ of the current associated with the $U(1)$ symmetry:

$$J^{\mu} \,=\, i\left(\phi^{\dagger}\,(\partial^{\mu}\phi) \,-\, (\partial^{\mu}\phi^{\dagger})\,\phi\right)$$

We begin with the parity operator:

$$\mathcal{P}\,J^{\mu}(t,\mathbf{x})\,\mathcal{P} \,=\, i\left(\phi^{\dagger}(t,-\mathbf{x})\,\partial^{\mu}(\phi(t,-\mathbf{x})) \,-\, \partial^{\mu}(\phi^{\dagger}(t,-\mathbf{x}))\,\phi(t,-\mathbf{x})\right)$$

Now,

$$\partial^{\mu}(\phi(t,-\mathbf{x})) \,=\, \eta^{\mu\mu}\,(\partial^{\mu}\phi)(t,-\mathbf{x})$$

Thus,

$$\mathcal{P}\,J^{\mu}(t,\mathbf{x})\,\mathcal{P} \,=\, \eta^{\mu\mu}\,J^{\mu}(t,-\mathbf{x})$$

For time reversal:

$$\mathcal{T}\,J^{\mu}(t,\mathbf{x})\,\mathcal{T} \,=\, -i\left(\phi^{\dagger}(-t,\mathbf{x})\,\partial^{\mu}(\phi(-t,\mathbf{x})) \,-\, \partial^{\mu}(\phi^{\dagger}(-t,\mathbf{x}))\,\phi(-t,\mathbf{x})\right)$$

Now,

$$\partial^{\mu}(\phi(-t,\mathbf{x})) \,=\, -\eta^{\mu\mu}\,(\partial^{\mu}\phi)(t,-\mathbf{x})$$

Thus,

$$\mathcal{T}\,J^{\mu}(t,\mathbf{x})\,\mathcal{T} \,=\, \eta^{\mu\mu}\,J^{\mu}(-t,\mathbf{x})$$

For charge conjugation:

$$\mathcal{C}\,J^{\mu}\,\mathcal{C} \,=\, i\left(\phi\,(\partial^{\mu}\phi^{\dagger}) \,-\, (\partial^{\mu}\phi)\,\phi^{\dagger}\right) \,=\, -\,J^{\mu}$$

This may be seen since J^{μ} is normal ordered and it may be seen that:

$$:\phi^{\dagger}(x)\,\phi(y): \,=\, :\phi(y)\,\phi^{\dagger}(x):$$

(c)

Any Hermitian Lorentz scalar built from ψ and ϕ may be written as a sum of products of Hermitian tensors, and their derivatives, which are quadratic in the fields. We consider the transformation properties of such tensors for the complex scalar and Dirac spinor fields under $\Theta = \mathcal{CPT}$. For the Dirac field we consider:

$$\bar{\psi}\psi \qquad i\bar{\psi}\gamma^5\psi \qquad \bar{\psi}\gamma^\mu\psi \qquad \bar{\psi}\gamma^\mu\gamma^5\psi \qquad \bar{\psi}\sigma^{\mu\nu}\psi$$

for the complex scalar field we consider:

$$\phi^\dagger\phi \qquad J^\mu = i\left(\phi^\dagger(\partial^\mu\phi) - (\partial^\mu\phi^\dagger)\,\phi\right) \qquad \partial^\mu\phi^\dagger\,\partial^\nu\phi$$

As can be seen from the results of part (b) and the table on pg 71 of P+S , all tensors with an even(odd) number of indices are even(odd) under Θ . Furthermore the derivative operator ∂_μ is odd under Θ . Thus, since all scalars are a contraction of an even number of indices, all Hermitian Lorentz scalars are invariant under Θ .