

QFT 1 : Problem Set 4

1.) Peskin & Schroeder 3.5 : Supersymmetry

In the interest of notational consistency with problem 2, we will adopt conventions that are common in treatments of supersymmetry. This will require us to rewrite expressions appearing in P&S. The following conventions are similar to those appearing in Ryder and in Bailin and Love. Wess and Bagger also use similar conventions except for the sign of the metric. There is a review by Lykken (hep-th/9612114) that uses somewhat different conventions. I highly recommend it for aspiring string theorists since it discusses dimensions other than four and places emphasis on formalism rather than phenomenology.

We represent Dirac spinors as:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

The γ matrices are given by:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

Thus the σ matrices have the following index structure:

$$(\sigma^\mu)_{\beta\dot{\alpha}} \quad (\bar{\sigma}^\mu)^{\dot{\beta}\alpha}$$

We raise undotted indices and lower dotted indices as follows:

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta \quad \bar{\chi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\chi}^{\dot{\beta}}$$

Note that we act with the second index of the ϵ tensor. We introduce inverses to raise dotted indices and lower undotted indices:

$$\epsilon^{\alpha\beta} \epsilon_{\beta\gamma} = \delta^\alpha_\gamma \quad \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}^{\dot{\gamma}}$$

Note that it is not consistent to raise or lower indices on the ϵ tensor itself since:

$$\epsilon^{\alpha\beta} \epsilon^{\gamma\delta} \epsilon_{\beta\delta} = \epsilon^{\gamma\alpha} = -\epsilon^{\alpha\gamma}$$

This problem can be easily resolved (see Wald (1984)) but we will stick with the conventions as they appear in most of the literature and avoid raising and lowering indices on the ϵ tensor.

We use the following explicit forms for the σ and ϵ tensors:

$$\sigma^\mu \equiv (\mathbf{1}, \boldsymbol{\sigma}) \quad \bar{\sigma}^\mu \equiv (\mathbf{1}, -\boldsymbol{\sigma})$$

And,

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = -i\sigma^2 \quad \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = i\sigma^2$$

We introduce the following conventions for Lorentz invariant quantities:

$$\chi\psi \equiv \chi^\alpha \psi_\alpha = -\psi_\alpha \chi^\alpha = \psi^\alpha \chi_\alpha = \psi\chi$$

Where we have used the Grassmann character of the fields and the antisymmetry of ϵ .

Similarly,

$$\bar{\chi}\bar{\psi} \equiv \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} = \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \bar{\psi}\bar{\chi}$$

We define complex conjugation to act as:

$$(\psi_\alpha)^* \equiv \bar{\psi}_{\dot{\alpha}} \quad (\bar{\chi}^{\dot{\alpha}})^* \equiv \chi^\alpha$$

With the following action on products of Grassmann variables:

$$(\psi^\alpha \chi_\beta)^* \equiv \bar{\chi}_\beta \bar{\psi}^{\dot{\alpha}}$$

We also define the Lorentz vector quantities:

$$\chi \sigma^\mu \bar{\psi} \equiv \chi^\alpha (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} = -\bar{\psi}_{\dot{\beta}} (\bar{\sigma}^\mu)^{\dot{\beta}\alpha} \chi_\alpha = -\bar{\psi} \bar{\sigma}^\mu \chi$$

Where we have used:

$$\epsilon^{\dot{\beta}\dot{\alpha}} (\sigma^\mu)_{\beta\dot{\alpha}} \epsilon^{\beta\alpha} = -(\bar{\sigma}^\mu)^{\dot{\beta}\alpha} \quad \text{or} \quad \sigma^2 (\sigma^\mu)^T \sigma^2 = \bar{\sigma}^\mu$$

I will not provide a detailed description of the Lorentz transformation properties of the various spinor types. The reader is encouraged to consult any of the references mentioned above or in the homework statement. Further conventions and identities will be introduced as needed.

(a)

We write the Lagrangian appearing in P&S in the above conventions:

$$\mathcal{L} = \partial\phi^* \cdot \partial\phi + i \bar{\chi} \bar{\sigma} \cdot \partial\chi + F^* F$$

P&S writes the supersymmetry transformation as:

$$\begin{aligned} \delta\phi &= -i \epsilon^T \sigma^2 \chi \\ \delta\chi &= \epsilon F + \sigma \cdot \partial\phi \sigma^2 \epsilon^* \\ \delta F &= -i \epsilon^\dagger \bar{\sigma} \cdot \partial\chi \end{aligned}$$

These are written in the above conventions as:

$$\begin{aligned} \delta\phi &= \epsilon \chi \\ \delta\chi_\alpha &= \epsilon_\alpha F - i (\sigma^\mu)_{\alpha\dot{\alpha}} \bar{\epsilon}^{\dot{\alpha}} \partial_\mu \phi \\ \delta F &= -i \bar{\epsilon} \bar{\sigma} \cdot \partial\chi \end{aligned}$$

We consider each term in the Lagrangian separately. We begin with:

$$\mathcal{L}_\phi = \partial\phi^* \cdot \partial\phi$$

Under the supersymmetry transformation:

$$\delta\mathcal{L}_\phi = \partial(\delta\phi)^* \cdot \partial\phi + \partial\phi^* \cdot \partial(\delta\phi)$$

Now,

$$(\delta\phi)^* = (\epsilon^\alpha \chi_\alpha)^* = \bar{\chi}_{\dot{\alpha}} \bar{\epsilon}^{\dot{\alpha}} = \bar{\epsilon} \bar{\chi}$$

Thus,

$$\delta\mathcal{L}_\phi = \bar{\epsilon} \partial\bar{\chi} \cdot \partial\phi + \partial\phi^* \cdot \partial\chi \epsilon$$

We now consider the supersymmetry transformation of:

$$\mathcal{L}_\chi = i \bar{\chi} \bar{\sigma} \cdot \partial\chi = i \bar{\chi}_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \partial_\mu \chi_\beta$$

We have:

$$\delta\mathcal{L}_\chi = i \delta\bar{\chi}_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \partial_\mu \chi_\beta + i \bar{\chi}_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \partial_\mu (\delta\chi_\beta)$$

Now,

$$\delta\bar{\chi}_{\dot{\alpha}} = \bar{\epsilon}_{\dot{\alpha}} F^* + i ((\sigma^\mu)_{\dot{\alpha}\alpha})^* \epsilon^\alpha \partial_\mu \phi^*$$

Where we have renamed indices on σ^μ to comport with the convention for the complex conjugate. Now, since $(\sigma^\mu)^\dagger = \sigma^\mu$:

$$((\sigma^\mu)_{\dot{\alpha}\alpha})^* = (\sigma^\mu)_{\alpha\dot{\alpha}}$$

Thus,

$$\delta\bar{\chi}_{\dot{\alpha}} = \bar{\epsilon}_{\dot{\alpha}} F^* + i \epsilon^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu \phi^*$$

And,

$$\delta\mathcal{L}_\chi = i \bar{\epsilon} (\bar{\sigma} \cdot \partial\chi) F^* - \epsilon \sigma^\mu (\bar{\sigma} \cdot \partial\chi) \partial_\mu \phi^* + i \bar{\chi} \bar{\sigma}^\mu \epsilon \partial_\mu F + \bar{\chi} \bar{\sigma}^\mu \sigma^\nu \bar{\epsilon} \partial_\mu \partial_\nu \phi$$

We now consider:

$$\mathcal{L}_F = F^* F \quad \text{so that} \quad \delta\mathcal{L}_F = \delta F^* F + F^* \delta F$$

Now,

$$\delta F^* = i (\partial_\mu \bar{\chi}_{\dot{\alpha}}) (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \epsilon_\alpha = i (\partial_\mu \bar{\chi}) \bar{\sigma}^\mu \epsilon$$

Thus,

$$\delta\mathcal{L}_F = i (\partial_\mu \bar{\chi}) \bar{\sigma}^\mu \epsilon F - i F^* \bar{\epsilon} \bar{\sigma} \cdot \partial\chi$$

Returning to $\delta\mathcal{L}_\chi$, we integrate the second term by parts and use the identities,

$$(\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu) = (\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu) = 2\eta^{\mu\nu}$$

to find:

$$\delta\mathcal{L}_\chi = -\partial_\nu (\epsilon \sigma^\mu \bar{\sigma}^\nu \chi \partial_\mu \phi^*) + i F^* \bar{\epsilon} \bar{\sigma} \cdot \partial\chi + i \bar{\chi} \bar{\sigma}^\mu \epsilon \partial_\mu F + \epsilon \chi \partial^2 \phi^* + \bar{\chi} \bar{\epsilon} \partial^2 \phi$$

Considering $\delta\mathcal{L}_\phi$, we integrate both terms by parts:

$$\delta\mathcal{L}_\phi = \partial_\mu (\bar{\chi} \bar{\epsilon} \partial^\mu \phi + \epsilon \chi \partial^\mu \phi^*) - \epsilon \chi \partial^2 \phi^* - \bar{\chi} \bar{\epsilon} \partial^2 \phi$$

We integrate the first term of $\delta\mathcal{L}_F$ by parts:

$$\delta\mathcal{L}_F = \partial_\mu (i \bar{\chi} \bar{\sigma}^\mu \epsilon F) - i \bar{\chi} \bar{\sigma}^\mu \epsilon \partial_\mu F - i F^* \bar{\epsilon} \bar{\sigma} \cdot \partial\chi$$

Thus, collecting terms, we find that $\delta\mathcal{L}$ is a total divergence:

$$\delta\mathcal{L} = \partial_\nu (\bar{\chi} \bar{\epsilon} \partial^\nu \phi + \epsilon \chi \partial^\nu \phi^* + i \bar{\chi} \bar{\sigma}^\nu \epsilon F - \epsilon \sigma^\mu \bar{\sigma}^\nu \chi \partial_\mu \phi^*)$$

(b)

We now introduce the following additional term to the Lagrangian:

$$\Delta\mathcal{L} = m(\phi F + \phi^* F^*) - \frac{m}{2}(\chi\chi + \bar{\chi}\bar{\chi})$$

Where we have transcribed the notation of P&S into that introduced above.

We define $\Delta\mathcal{L} = \Omega + \Omega^*$ where:

$$\Omega = m\phi F - \frac{m}{2}\chi\chi$$

Under the supersymmetry transformation we have:

$$\delta\Omega = m\phi\delta F + m\delta\phi F - m\chi\delta\chi$$

Thus,

$$\delta\Omega = -im\phi\bar{\epsilon}\bar{\sigma}\cdot\partial\chi + im\chi\sigma^\mu\bar{\epsilon}\partial_\mu\phi$$

Or, using $\chi \sigma^\mu \bar{\epsilon} = -\bar{\epsilon} \bar{\sigma}^\mu \chi$:

$$\delta\Omega = \partial_\mu (-im\phi \bar{\epsilon} \bar{\sigma}^\mu \chi)$$

Thus, up to a total divergence, $\Delta\mathcal{L}$ is invariant under the supersymmetry transformation. We now write out the augmented Lagrangian:

$$\mathcal{L} + \Delta\mathcal{L} = \partial\phi^* \cdot \partial\phi + i\bar{\chi}\bar{\sigma} \cdot \partial\chi + F^*F + m(\phi F + \phi^*F^*) - \frac{m}{2}(\chi\chi + \bar{\chi}\bar{\chi})$$

Using the equations of motion for F ,

$$F^* + m\phi = 0$$

We rewrite the Lagrangian as:

$$\mathcal{L} + \Delta\mathcal{L} = \partial\phi^* \cdot \partial\phi - m^2\phi^*\phi + i\bar{\chi}\bar{\sigma} \cdot \partial\chi - \frac{m}{2}(\chi\chi + \bar{\chi}\bar{\chi})$$

This is the Lagrangian for a complex scalar field and a Majorana spinor field of equal mass. Please see P&S problem 3.4.

(c)

We now consider a Lagrangian which contains the fields (ϕ_i, χ_i, F_i) for $i = 1 \dots n$:

$$\mathcal{L} = \partial\phi_i^* \cdot \partial\phi_i + i\bar{\chi}_i\bar{\sigma} \cdot \partial\chi_i + F_i^*F_i + \Gamma + \Gamma^*$$

Where,

$$\Gamma = F_i W_i - \frac{1}{2} W_{ij} \chi_i \chi_j$$

Where we introduce a superpotential $W[\phi]$ and define:

$$W_{i_1 \dots i_p} \equiv \frac{\partial W[\phi]}{\partial \phi_{i_1} \dots \phi_{i_p}}$$

For instance, the Lagrangian in part (b) corresponds to $W[\phi] = m\phi^2/2$. We have already shown in part (a) that the first three terms yield a total derivative under the supersymmetry transformation:

$$\begin{aligned} \delta\phi_i &= \epsilon \chi_i \\ (\delta\chi_i)_\alpha &= \epsilon_\alpha F_i - i(\sigma^\mu)_{\alpha\dot{\alpha}} \bar{\epsilon}^{\dot{\alpha}} \partial_\mu \phi_i \\ \delta F_i &= -i\bar{\epsilon}\bar{\sigma} \cdot \partial\chi_i \end{aligned}$$

Note that we use the same Grassmann parameter ϵ for each component of the respective fields. Thus to prove that \mathcal{L} is supersymmetric we examine:

$$\delta\Gamma = \delta F_i W_i + F_i W_{ij} \delta\phi_j - W_{ij} \chi_i \delta\chi_j - \frac{1}{2} W_{ijk} \chi_i \chi_j \delta\phi_k$$

Or,

$$\begin{aligned} \delta\Gamma &= -i\bar{\epsilon}\bar{\sigma} \cdot \partial\chi_i W_i + F_i W_{ij} \epsilon \chi_j - W_{ij} \chi_i \epsilon F_j \\ &+ i W_{ij} \chi_i \sigma^\mu \bar{\epsilon} \partial_\mu \phi_j - \frac{1}{2} W_{ijk} \chi_i \chi_j \epsilon \chi_k \end{aligned}$$

From the symmetry of W_{ij} :

$$W_{ij} (F_i \chi_j \epsilon - F_j \chi_i \epsilon) = 0$$

Also, using $\chi_i \sigma^\mu \bar{\epsilon} = -\bar{\epsilon} \bar{\sigma}^\mu \chi_i$:

$$-i\bar{\epsilon}\bar{\sigma} \cdot \partial\chi_i W_i = i(\partial_\mu \chi_i) \sigma^\mu \bar{\epsilon} W_i = \partial_\mu (i\chi_i \sigma^\mu \bar{\epsilon} W_i) - i W_{ij} \chi_i \sigma^\mu \bar{\epsilon} \partial_\mu \phi_j$$

Thus,

$$\delta\Gamma = \partial_\mu (i \chi_i \sigma^\mu \bar{\epsilon} W_i) - \frac{1}{2} W_{ijk} \chi_i \chi_j \epsilon \chi_k$$

We could now introduce a Fierz identity but it is simpler to define $\chi_i = (a_i, b_i)$ and write:

$$\chi_i \chi_j \epsilon \chi_k = (b_i a_j - a_i b_j) (\epsilon_2 a_k - \epsilon_1 b_k)$$

Upon expansion, each element in this product is anti-symmetric under exchange of one pair of indices. Since W_{ijk} is totally symmetric, the second term in $\delta\Gamma$ vanishes and we have:

$$\delta\Gamma = \partial_\mu (i \chi_i \sigma^\mu \bar{\epsilon} W_i)$$

Thus \mathcal{L} is supersymmetric.

We now consider $n = 1$ and $W[\phi] = g\phi^3/3$:

$$\mathcal{L} = \partial\phi^* \cdot \partial\phi + i \bar{\chi} \bar{\sigma} \cdot \partial\chi + F^* F + \Gamma + \Gamma^*$$

Where,

$$\Gamma = F g \phi^2 - g \phi \chi \chi$$

The equation of motion for F yields:

$$F^* + g \phi^2 = 0$$

Thus, after some algebra:

$$\mathcal{L} = \partial\phi^* \cdot \partial\phi - g^2 (\phi^* \phi)^2 + i \bar{\chi} \bar{\sigma} \cdot \partial\chi - g (\phi \chi \chi + \phi^* \bar{\chi} \bar{\chi})$$

This leads to the following equation of motion for ϕ :

$$\partial^2 \phi + 2g^2 \phi (\phi^* \phi) + g \phi^* \bar{\chi} \bar{\chi} = 0$$

And for χ :

$$i \bar{\sigma} \cdot \partial\chi - 2g \phi^* \bar{\chi} = 0$$

2.)

(a)

We introduce the superspace representation of the supersymmetry generators:

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu)_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu \quad \text{and} \quad \bar{Q}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu$$

To find the anticommutator algebra of Q_α and $\bar{Q}_{\dot{\alpha}}$ we must compute a total of 10 terms. First we have, trivially:

$$\left\{ \frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\theta^\beta} \right\} = \left\{ \frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \right\} = \left\{ \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} \right\} = 0$$

Defining,

$$A_\alpha = i(\sigma^\mu)_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu \quad \text{and} \quad \bar{A}_{\dot{\alpha}} = i\theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu$$

And using,

$$\{\theta^\alpha, \theta^\beta\} = \{\theta^\alpha, \bar{\theta}^{\dot{\alpha}}\} = \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = 0$$

We find:

$$\{A_\alpha, A_\beta\} = \{A_\alpha, \bar{A}_{\dot{\alpha}}\} = \{\bar{A}_{\dot{\alpha}}, \bar{A}_{\dot{\beta}}\} = 0$$

Also since,

$$\left\{ \frac{\partial}{\partial\theta^\alpha}, \bar{\theta}^{\dot{\alpha}} \right\} = \left\{ \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \theta^\alpha \right\} = 0$$

We find:

$$\left\{ \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, A_\beta \right\} = \left\{ \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \bar{A}_{\dot{\beta}} \right\} = 0$$

Finally, since,

$$\left\{ \frac{\partial}{\partial\theta^\alpha}, \theta^\beta \right\} = \delta_\alpha^\beta \quad \text{and} \quad \left\{ \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \bar{\theta}^{\dot{\beta}} \right\} = \delta^{\dot{\beta}}_{\dot{\alpha}}$$

We find:

$$\left\{ \frac{\partial}{\partial\theta^\alpha}, \bar{A}_{\dot{\beta}} \right\} = \left\{ \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, A_\beta \right\} = i(\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu$$

The above relations lead to:

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \quad \text{and} \quad \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -2i(\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu$$

Introducing the derivative operators:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + A_\alpha \quad \text{and} \quad \bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \bar{A}_{\dot{\alpha}}$$

We find:

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \quad \text{and} \quad \{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2i(\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu$$

And,

$$\{Q_\alpha, D_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = \{Q_\alpha, \bar{D}_{\dot{\alpha}}\} = \{D_\alpha, \bar{Q}_{\dot{\alpha}}\} = 0$$

(b)

The super-derivatives:

$$D_{\pm} = \frac{\partial}{\partial \theta^{\pm}} + i \theta^{\pm} \partial_{\pm}$$

Let us start by writing the super-charges:

$$Q_{\pm} = \frac{\partial}{\partial \theta^{\pm}} - i \theta^{\pm} \partial_{\pm}$$

One can easily verify that the above definition of supercharges satisfy the correct algebra:

$$\begin{aligned} \{Q_{\pm}, Q_{\pm}\} &= -2i\theta^{\pm} \partial_{\pm} \\ \{Q_{+}, Q_{-}\} &= 0 \\ \{Q_{\pm}, D_{\pm}\} &= 0 \\ \{Q_{+}, D_{-}\} &= 0 \end{aligned}$$

The 1 + 1 d super-field is:

$$\Phi(x_{\pm}, \theta_{\pm}) = \phi(x) + \theta^{\pm} \chi_{\pm}(x) + \theta^{+} \theta^{-} F(x)$$

The given SUSY transformations are:

$$\begin{aligned} \delta \phi &= \epsilon^{\pm} \chi_{\pm} \\ \delta \chi_{\pm} &= \epsilon^{\mp} F - i \epsilon^{\pm} \partial_{\pm} \phi \\ \delta F &= -i \epsilon^{\pm} \partial_{\pm} \chi_{\mp} \end{aligned}$$

Let us consider:

$$\delta \Phi = [\epsilon^{\pm} Q_{\pm}, \Phi] = [\epsilon^{+} Q_{+}, \Phi] + [\epsilon^{-} Q_{-}, \Phi]$$

We know that:

$$\begin{aligned} [\epsilon^{+} Q_{+}, \Phi] &= \left[\epsilon^{+} \frac{\partial}{\partial \theta^{+}}, \Phi \right] - i [\epsilon^{+} \theta^{+} \partial_{+}, \Phi] \\ &= \epsilon^{+} \chi_{+} + \epsilon^{+} \theta^{-} F - i \epsilon^{+} \theta^{+} \partial_{+} \phi - i \epsilon^{+} \theta^{+} \theta^{-} \partial_{+} \chi_{-} \end{aligned}$$

We may similarly evaluate $[\epsilon^{-} Q_{-}, \Phi]$ and add the 2 contributions to obtain:

$$\delta \Phi = \epsilon^{\pm} \chi_{\pm} + \theta^{\pm} (i \epsilon^{\pm} \partial_{\pm} \phi \pm \epsilon^{\mp} F) + \theta^{+} \theta^{-} (\mp i \epsilon^{\pm} \partial_{\pm} \chi_{\mp})$$

Comparing with the transformation of the 1 + 1 d super-field:

$$\delta \Phi = \delta \phi + \theta^{\pm} \delta \chi_{\pm} + \theta^{+} \theta^{-} \delta F$$

We find:

$$\begin{aligned} \delta \phi &= \epsilon^{\pm} \chi_{\pm} \\ \delta \chi_{\pm} &= \pm \epsilon^{\mp} F + i \epsilon^{\pm} \partial_{\pm} \phi \\ \delta F &= \pm i \epsilon^{\pm} \partial_{\pm} \chi_{\mp} \end{aligned}$$

This matches the given SUSY transformations up to a few signs.

Let us now take a look at the SUSY Lagrangian and check whether it reduces down to the expected 1 + 1 d form:

$$\mathcal{L} = \int d^2x d^2\theta (D_+\Phi D_-\Phi + W[\Phi])$$

We know that:

$$\begin{aligned} D_+\Phi &= \left(\frac{\partial}{\partial\theta^+} + i\theta^+\partial_+ \right) (\phi + \theta^\pm\chi_\pm + \theta^+\theta^-F) \\ &= \chi_+ + \theta^-F + i\theta^+\partial_+\phi + i\theta^+\theta^-\partial_+\chi_- \\ D_-\Phi &= \left(\frac{\partial}{\partial\theta^-} + i\theta^-\partial_- \right) (\phi + \theta^\pm\chi_\pm + \theta^+\theta^-F) \\ &= \chi_- - \theta^+F + i\theta^-\partial_-\phi - i\theta^+\theta^-\partial_-\chi_+ \end{aligned}$$

Since the θ s anti-commute with each other $\theta^2 = 0$, and the Kinetic part of the Lagrangian simplifies to:

$$\begin{aligned} \mathcal{L}_K &= \int d^2x d^2\theta (D_+\Phi D_-\Phi) \\ &= \int d^2x d\theta^+ d\theta^- [\chi_+\chi_- + \theta^\pm (\chi_\pm F \pm i\chi_\mp\partial_\pm\phi)] \\ &\quad + \int d^2x d\theta^+ d\theta^- \theta^+\theta^- (i\chi_\pm\partial_\mp\chi_\pm + F^2 - \partial_+\phi\partial_-\phi) \end{aligned}$$

We now make use of the following properties of Grassmann variable θ :

$$\begin{aligned} \int d\theta^+ d\theta^- &= 0 \\ \int d\theta^+ d\theta^- \theta^\pm &= 0 \\ \int d\theta^+ d\theta^- \theta^+\theta^- &= -1 \end{aligned}$$

The Lagrangian then simplifies to:

$$\mathcal{L}_K = \int d^2x (\partial_+\phi\partial_-\phi - i\chi_\pm\partial_\mp\chi_\pm - F^2)$$

which is the expected form of the Kinetic Part of the 1 + 1 d Lagrangian.

Now let us move on to check the interaction term and expand it to second order in θ keeping in mind that only the term having $\theta^+\theta^-$ survives because of the integral property of Grassmann variable θ :

$$\begin{aligned} \mathcal{L}_I &= \int d^2x d^2\theta W[\Phi] \\ &= \int d^2x \frac{1}{2} \frac{\partial^2 W}{\partial\theta^-\partial\theta^+} \Big|_{\theta^\pm=0} \end{aligned}$$

We now evaluate the partial derivatives as follows:

$$\begin{aligned} \frac{\partial W}{\partial\theta^+} &= \frac{\partial W}{\partial\Phi} (\chi_+ + \theta^-F) \\ \frac{\partial^2 W}{\partial\theta^-\partial\theta^+} \Big|_{\theta^\pm=0} &= \frac{\partial W}{\partial\phi} F + \frac{\partial^2 W}{\partial^2\phi} \chi_-\chi_+ \end{aligned}$$

Putting all of this together we finally arrive at the expected form of the complete 1 + 1 d lagrangian which we may now write as follows:

$$\begin{aligned}\mathcal{L} &= \int d^2x (\partial_+ \phi \partial_- \phi - i\chi_\pm \partial_\mp \chi_\pm - F^2) \\ &+ \int d^2x \frac{1}{2} \left(\frac{\partial W}{\partial \phi} F + \frac{\partial^2 W}{\partial \phi^2} \chi_- \chi_+ \right)\end{aligned}$$

3.) Peskin & Schroeder 9.2 (d)

We begin with the following Euclidean Lagrangian for a Fermionic oscillator where $\psi(t)$, $\bar{\psi}(t)$ are Grassmann fields:

$$L_E = \bar{\psi} \dot{\psi} + \omega \bar{\psi} \psi$$

We compute the partition function:

$$Z(\beta) = \int \mathcal{D}\bar{\psi}_0 \mathcal{D}\psi_0 \int_{\psi(0)=\psi_0}^{\psi(\beta)=-\psi_0} \int \mathcal{D}\bar{\psi}(t) \mathcal{D}\psi(t) \exp \left[- \int_0^\beta dt L_E[\bar{\psi}, \psi] \right]$$

We first introduce an expansion for ψ in terms of anti-periodic functions:

$$\psi(t) = \frac{1}{\sqrt{\beta}} \sum_n x_n e^{2\pi i(n+1/2)t/\beta} \quad \text{and} \quad \bar{\psi}(t) = \frac{1}{\sqrt{\beta}} \sum_n \bar{x}_n e^{-2\pi i(n+1/2)t/\beta}$$

Computing the Euclidean action, we find:

$$\int_0^\beta dt L_E[\bar{\psi}, \psi] = \sum_n \Omega_n \bar{x}_n x_n \quad \text{where} \quad \Omega_n = (2\pi i(n+1/2)/\beta + \omega)$$

We make the substitution:

$$\int \mathcal{D}\bar{\psi}_0 \mathcal{D}\psi_0 \int_{\psi(0)=\psi_0}^{\psi(\beta)=-\psi_0} \int \mathcal{D}\bar{\psi}(t) \mathcal{D}\psi(t) = \prod_n \int d\bar{x}_n dx_n$$

Thus,

$$Z = \prod_n \int d\bar{x}_n dx_n e^{-\Omega_n \bar{x}_n x_n} = \prod_n \int d\bar{x}_n dx_n (1 - \Omega_n \bar{x}_n x_n) = \prod_n \Omega_n$$

Where we have used the fact that x_n, \bar{x}_n are Grassmann variables so that:

$$\{x_n, x_m\} = \{x_n, \bar{x}_m\} = \{\bar{x}_n, \bar{x}_m\} = 0$$

And,

$$\int d\bar{x}_n dx_n = 0 \quad \text{and} \quad \int d\bar{x}_n dx_n x_n \bar{x}_n = 1$$

Defining $\alpha = (i\pi/2 + \beta\omega/2)$ so that $\Omega_n = 2/\beta(i\pi n + \alpha)$, we find:

$$Z = \Omega_n \prod_{n=1}^{\infty} \Omega_n \Omega_{-n} = \left[\prod_n (2/\beta) \right] \alpha \prod_{n=1}^{\infty} ((\pi n)^2 + \alpha^2)$$

Using the product representation of $\sinh(z)$ given in P+S:

$$Z = \left[(2i/\beta) \prod_{n=1}^{\infty} (2\pi n\beta)^2 \right] (-i \sinh(\alpha))$$

Thus, dropping the infinite ω -independent factor in brackets:

$$Z(\beta) = -i \sinh(\alpha) = \cosh(\beta\omega/2)$$