QFT 1 : Problem Set 5

1.) Peskin & Schroeder 4.1

We begin with the Shroedinger picture Hamiltonian for a real Klein-Gordon field interacting with an external source:

$$H_S(t) \;=\; H_0 \;-\; \int d^3x\, j(\mathbf{x},t)\,\phi(\mathbf{x})$$

Here H_0 is the normal-ordered free Klein-Gordon Hamiltonian and $\phi(\mathbf{x})$ is the field operator in the Shroedinger picture. That is,

$$\phi(\mathbf{x}) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\left(\,a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} \;+\; a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\,\right)$$

And,

$$H_0 \;=\; \int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\,a^\dagger_{\mathbf{p}}\,a_{\mathbf{p}}$$

We consider a state vector in the interaction picture:

$$|\Psi_I(t)\rangle \;=\; e^{iH_0t}\;|\Psi_S(t)\rangle$$

From the Schroedinger equation we find:

$$i\,\partial_t\,|\Psi_I(t)\rangle \;=\; H_I(t)\,|\Psi_I(t)\rangle$$

Where,

$$H_I(t) \;=\; e^{iH_0t}\,(\,H_S(t) \;-\; H_0\,)\,e^{-iH_0t} \;=\; -\int d^3x\, j(\mathbf{x},t)\,\phi_I(\mathbf{x},t)$$

And,

$$\phi_I(x) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\left(\,a_{\mathbf{p}}e^{-ip\cdot x} \;+\; a^\dagger_{\mathbf{p}}e^{ip\cdot x}\,\right)$$

The equation of motion has the formal solution:

$$|\Psi_I(t_2)\rangle \;=\; U(t_2,t_1)\;|\Psi_I(t_1)\rangle$$

Where, introducing time-ordering, we have:

$$U(t_2,t_1) \;=\; T\left[\exp\left(-i\int_{t_1}^{t_2} dt\,H_I(t)\right)\right]$$

(a)

We consider the amplitude for the system to evolve from the Schroedinger picture vacuum $|\,\Omega_S(-\tau)\rangle$ at time $-\tau$ to the Schroedinger picture vacuum $|\,\Omega_S(\tau)\rangle$ at time τ in the limit $\tau \to \infty$. In the interaction picture this is given by:

$$A(0) \;=\; \lim_{\tau\to\infty}\,\langle\,\Omega_I(\tau)|\;U(\tau,-\tau)\;|\,\Omega_I(-\tau)\rangle$$

The Schroedinger picture vacuum is defined by:

$$H_S(t)\,|\,\Omega_S(t)\rangle \;=\; 0 \qquad\text{and}\qquad \langle\,\Omega_S(t)|\;\Omega_S(t)\rangle \;=\; 1$$

We assume that $j(\mathbf{x},t)=0$ for $|t|\geq\tau$. Thus,

$$H_S(\pm\tau) = H_0 \qquad \text{and} \qquad |\,\Omega_S(\pm\tau)\rangle = |0\rangle$$

Here $|0\rangle$ is the free Klein-Gordon vacuum defined by:

$$H_0\,|0\rangle \;=\; 0 \qquad \text{and} \qquad \langle 0|\,0\rangle \;=\; 1$$

Thus,

$$|\,\Omega_I(\pm\tau)\rangle \;=\; e^{\pm iH_0 t}\;|0\rangle \;=\; |0\rangle$$

This leads to:

$$A(0) \;=\; \langle 0|\;T\left[\exp\left(\,i\int d^4x\, j(x)\,\phi_I(x)\right)\right]|0\rangle$$

Thus the probability for vacuum to evolve to the vacuum is:

$$P(0) \;=\; |A(0)|^2 \;=\; \left|\langle 0|\;T\left[\exp\left(\,i\int d^4x\, j(x)\,\phi_I(x)\right)\right]|0\rangle\right|^2$$

In what follows we drop the I subscript and write $\phi_I(x)$ as $\phi(x)$. Also, where convenient, we write $\phi(x)$ as ϕ_x or $\phi(x_i)$ as ϕ_i. We also abbreviate $\int d^4x$ as $\int dx$.

(b)

We consider the expansion of $A(0)$ to $\mathcal{O}(j^2)$:

$$A_2(0) \;=\; \langle 0|\;T\left[1 \,+\, i\int dx\, j_x\,\phi_x \;-\; \tfrac{1}{2}\int dx\,dy\, j_x\, j_y\,\phi_x\,\phi_y\right]|0\rangle$$

Since $\langle 0|\,\phi_x\,|0\rangle = 0$, we have $A_2 = 1-\lambda/2$. Where we define:

$$\lambda \;=\; \int dx\,dy\, j_x\, j_y\,\langle 0|\,T\,(\phi_x\,\phi_y)\,|0\rangle$$

Since,

$$\langle 0|\,T\,(\phi_x\,\phi_y)\,|0\rangle \;=\; D_F(x-y) \;=\; \int \frac{d^4p}{(2\pi)^4}\,\frac{i\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}$$

We have:

$$\lambda \;=\; \int \frac{d^4p}{(2\pi)^4}\,\frac{i\,\tilde{\jmath}(p)\,\tilde{\jmath}(-p)}{p^2-m^2+i\epsilon} \qquad \text{where} \qquad \tilde{\jmath}(p) \;=\; \int d^4x\, e^{ip\cdot x}\, j(x)$$

To $\mathcal{O}(\epsilon)$, writing $\omega = p^0$ and $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2}$, we have:

$$p^2-m^2+i\epsilon \;=\; (\omega-\gamma_\epsilon)\,(\omega+\gamma_\epsilon) \qquad \text{where} \qquad \gamma_\epsilon \;=\; E_{\mathbf{p}} \;-\; i\epsilon/(2E_{\mathbf{p}})$$

Thus,

$$\lambda \;=\; \frac{i}{2\pi}\int \frac{d^3p}{(2\pi)^3}\int d\omega\,\frac{|\tilde{\jmath}(\omega,\mathbf{p})|^2}{(\omega-\gamma_\epsilon)\,(\omega+\gamma_\epsilon)}$$

Where we have used the reality condition $\tilde{\jmath}(-p) = (\tilde{\jmath}(p))^*$. Closing the ω contour in either half-plane, we find:

$$\lambda \;=\; \lambda^* \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{|\tilde{\jmath}(p)|^2}{2E_{\mathbf{p}}}$$

Thus, to $\mathcal{O}(j^2)$:

$$P_2(0) \;=\; |A_2(0)|^2 \;=\; 1 \,-\, (\lambda+\lambda^*)\,/2 \;=\; 1 \,-\, \lambda$$

(c)

We represent λ as a Feynman diagram with external sources:

$$\lambda \;=$$

$$= \int dx\, dy\, j_x\, j_y\, \langle 0|\, T\,(\phi_x\, \phi_y)\, |0\rangle$$

We now demonstrate that the vacuum to vacuum amplitude may be written as:

$$A(0) \;=\; e^{-\lambda/2} \;=\; \exp\big[-\tfrac{1}{2}$$

$$\big]$$

We first expand the exponential in $A(0)$:

$$\begin{aligned} A(0) &= \langle 0|\, T\left[\exp\left(i \int d^4x\, j(x)\, \phi(x)\right)\right] |0\rangle \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[\prod_{k=1}^{n} \int dx_k\, j_k\right] \langle 0|\, T\,(\phi_1 \ldots \phi_n)\, |0\rangle \end{aligned}$$

We now expand the time-ordered product using Wicks theorem. Terms in which n is odd vanish. For a given n there are $(n-1)!!$ terms, each of which is a product of $n/2$ Feynman propagators. Since we are integrating with products of the sources $j_1 \ldots j_n$, we may change variables to obtain $(n-1)!!$ identical terms:

$$\begin{aligned} A(0) &= \sum_{\substack{n=0 \\ even}}^{\infty} \frac{i^n}{n!}\, (n-1)!! \left(\int dx\, dy\, j_x\, j_y\, \langle 0|\, T\,(\phi_x\, \phi_y)\, |0\rangle\right)^{n/2} \\ &= \sum_{\substack{n=0 \\ even}}^{\infty} \frac{i^n}{n!}\, (n-1)!!\, \lambda^{n/2} \end{aligned}$$

Now, for $n = 2m$:

$$\frac{(n-1)!!}{n!} \;=\; \frac{1}{2^m\, m!}$$

Thus, as above,

$$A(0) \;=\; \sum_{m=0}^{\infty} \frac{(i)^{2m}}{m!} (\lambda/2)^m \;=\; e^{-\lambda/2}$$

(d)

We now consider:

$$A(\{\mathbf{k}\}) \;=\; \langle \mathbf{k}|\, T\left[\exp\left(i \int d^4x\, j(x)\, \phi(x)\right)\right] |0\rangle$$

We first compute this amplitude to $\mathcal{O}(j)$. Now, from $|\mathbf{k}\rangle = \sqrt{2E_{\mathbf{k}}}\, a^\dagger_{\mathbf{k}}\, |0\rangle$, we find:

$$\langle \mathbf{k}|\, 0\rangle \;=\; 0 \qquad \text{and} \qquad \langle \mathbf{k}|\; \phi(x)\; |0\rangle \;=\; e^{ik\cdot x}$$

Thus,

$$A_1(\{\mathbf{k}\}) \;=\; i \int d^4x\, j(x)\, e^{ik\cdot x} \;=\; \tilde{\jmath}(k)$$

We might consider the probabilty density in momentum space to be the square of this amplitude but this would treat the integration measure too naively. Rather we should first introduce the projection operator onto the single particle subspace of the full asymptotic Fock space:

$$\mathbf{1}_1 \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\;|\mathbf{p}\rangle\,\langle\mathbf{p}|$$

The probability of finding a single particle in the outgoing state is:

$$P(1) \;=\; \int d^3p\,P(\{\mathbf{p}\}) \;=\; \mathrm{tr}\,(\,\mathbf{1}_1\;|\mathrm{out}\rangle\,\langle\mathrm{out}|\,)$$

Here,

$$|\mathrm{out}\rangle \;\equiv\; S\,|0\rangle \qquad \text{where} \qquad S \;\equiv\; \lim_{\tau\to\infty} U(\tau,-\tau) \;=\; T\,e^{\,i\int dx\,j(x)\,\phi(x)}$$

Thus,

$$P(1) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\;|\langle\mathbf{p}|\,S\,|0\rangle|^2$$

Or,

$$P(\{\mathbf{p}\}) \;=\; \frac{|\langle\mathbf{p}|\,S\,|0\rangle|^2}{(2\pi)^3\,2E_{\mathbf{p}}} \;=\; \frac{|A(\{\mathbf{p}\})|^2}{(2\pi)^3\,2E_{\mathbf{p}}}$$

Thus to $\mathcal{O}(j)$:

$$P_1(\{\mathbf{p}\}) \;=\; \frac{|\tilde{\jmath}(p)|^2}{(2\pi)^3\,2E_{\mathbf{p}}}$$

We now consider this quantity to all orders in j:

$$A(\{\mathbf{p}\}) \;=\; \langle\mathbf{p}|\,S\,|0\rangle \;=\; \sum_{n=0}^{\infty}\frac{i^n}{n!}\left[\prod_{k=1}^{n}\int dx_k\,j_k\right]\langle\mathbf{p}|\,T\,(\phi_1\ldots\phi_n)\,|0\rangle$$

At a particular point in the $4n$ dimensional space being integrated over at order n, we have some permutation of the fields due to time-ordering:

$$\begin{aligned}
\langle\mathbf{p}|\,T\,(\phi_1\ldots\phi_n)\,|0\rangle \;&=\; \sqrt{E_{\mathbf{p}}}\;\langle 0|\,a_{\mathbf{p}}\,\phi_{\pi_1}\ldots\phi_{\pi_n}\,|0\rangle \\
&=\; \sqrt{E_{\mathbf{p}}}\;\langle 0|\,[a_{\mathbf{p}},\phi_{\pi_1}\ldots\phi_{\pi_n}]\,|0\rangle \\
&=\; \sqrt{E_{\mathbf{p}}}\sum_{j=1}^{n}\langle 0|\,\phi_{\pi_1}\ldots\phi_{\pi_{j-1}}[a_{\mathbf{p}},\phi_{\pi_j}]\phi_{\pi_{j+1}}\ldots\phi_{\pi_n}\,|0\rangle \\
&=\; \sqrt{E_{\mathbf{p}}}\sum_{j=1}^{n}\langle 0|\,T\left(\phi_{\pi_1}\ldots\phi_{\pi_{j-1}}\,\phi_{\pi_{j+1}}\ldots\phi_{\pi_n}\right)|0\rangle\;[a_{\mathbf{p}},\phi_{\pi_j}]
\end{aligned}$$

Here we have re-introduced the time-ordering since the ordering is preserved and removed the commutator since it is a c-number:

$$[a_{\mathbf{p}},\phi(x)] \;=\; \frac{1}{\sqrt{E_{\mathbf{p}}}}\,e^{ip\cdot x}$$

Since we are integrating with products of the sources $j_1\ldots j_n$, we may change variables to replace the index π_j with n:

$$\left[\prod_{k=1}^{n}\int dx_k\,j_k\right]\langle\mathbf{p}|\,T\,(\phi_1\ldots\phi_n)\,|0\rangle \;=\; n\,\tilde{\jmath}(p)\,\left[\prod_{k=1}^{n-1}\int dx_k\,j_k\right]\langle 0|\,T\,(\phi_1\ldots\phi_{n-1})\,|0\rangle$$

Thus, since $\langle \mathbf{k} | 0 \rangle = 0$, we have:

$$A(\{\mathbf{p}\}) = \sum_{n=1}^{\infty} \frac{i^n}{n!} n \, \tilde{\jmath}(p) \left[\prod_{k=1}^{n-1} \int dx_k \, j_k \right] \langle 0 | T(\phi_1 \dots \phi_{n-1}) | 0 \rangle$$

Or, changing the limits of the sum:

$$A(\{\mathbf{p}\}) = i \, \tilde{\jmath}(p) \, A(0) = i \, \tilde{\jmath}(p) \, e^{-\lambda/2}$$

Thus, to all orders in j:

$$P(\{\mathbf{p}\}) = e^{-\lambda/2} \frac{|\tilde{\jmath}(p)|^2}{(2\pi)^3 \, 2E_{\mathbf{p}}} \qquad \text{where} \qquad \lambda = \int \frac{d^3p}{(2\pi)^3} \frac{|\tilde{\jmath}(p)|^2}{2E_{\mathbf{p}}}$$

(e)

We now consider the projection operator onto the n-particle subspace of the Fock space:

$$\mathbf{1}_n = \frac{1}{n!} \left[\prod_{k=1}^{n} \int \frac{d^3p_k}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}_k}} \right] |\mathbf{p}_1 \dots \mathbf{p}_n \rangle \langle \mathbf{p}_1 \dots \mathbf{p}_n |$$

Here the $n!$ comes from the Bose symmetry of the scalar field. Also,

$$|\mathbf{p}_1 \dots \mathbf{p}_n \rangle = \left[\prod_{k=1}^{n} \sqrt{E_{\mathbf{p}_k}} \, a^\dagger_{\mathbf{p}_k} \right] |0\rangle$$

The probability to create n particles is:

$$P(n) = \langle 0 | S^\dagger \, \mathbf{1}_n \, S | 0 \rangle = \frac{1}{n!} \left[\prod_{k=1}^{n} \int \frac{d^3p_k}{(2\pi)^3} \right] \left| \langle 0 | \left[\textstyle\prod_{k=1}^{n} a_{\mathbf{p}_k} \right] S | 0 \rangle \right|^2$$

Using the result from part (d), we find:

$$\langle 0 | \left[\textstyle\prod_{k=1}^{n} a_{\mathbf{p}_k} \right] S | 0 \rangle = \langle 0 | S | 0 \rangle \prod_{k=1}^{n} \frac{i \, \tilde{\jmath}(p_k)}{\sqrt{E_{\mathbf{p}_k}}}$$

Thus,

$$P(n) = \frac{1}{n!} \left| \langle 0 | S | 0 \rangle \right|^2 \left(\int \frac{d^3p}{(2\pi)^3} \frac{|\tilde{\jmath}(p)|^2}{2E_{\mathbf{p}}} \right)^n = \frac{1}{n!} \lambda^n e^{-\lambda}$$

(f)

Some facts about the Poisson distribution:

$$\sum_{n=0}^{\infty} P(n) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n = 1$$

$$\langle N \rangle = \sum_{n=0}^{\infty} n \, P(n) = e^{-\lambda} \sum_{n=1}^{\infty} n \frac{1}{n!} \lambda^n = \lambda \, e^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n = \lambda$$

$$\langle N^2 \rangle = \sum_{n=0}^{\infty} n^2 \, P(n) = \lambda \, e^{-\lambda} \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \lambda^{n-1} = \lambda \, e^{-\lambda} \sum_{n=0}^{\infty} \frac{(n+1)}{n!} \lambda^n = \lambda (\lambda + 1)$$

Thus,

$$\left\langle (N - \langle N \rangle)^2 \right\rangle = \langle N^2 \rangle - (\langle N \rangle)^2 = \lambda$$

2.) Peskin & Schroeder 4.2

We begin with the Lagrangian:

$$\mathcal{L} \,=\, \tfrac{1}{2}\left[\,\partial\Phi\cdot\partial\Phi \,-\, M^2\Phi^2\,\right] \,+\, \tfrac{1}{2}\left[\,\partial\phi\cdot\partial\phi \,-\, m^2\phi^2\,\right] \,-\, \mu\,\Phi\,\phi^2$$

We consider the rate at which a Φ particle decays into two ϕ particles to first order in μ.
We have the (connected and amputated) amplitude:

$$(2\pi)^4\,\delta^4\,(\,p_0 - (p_1 + p_2))\; i\,\mathcal{M} \,=\, \langle\,\mathbf{p}_1\,\mathbf{p}_2\,|\,T\left(e^{-i\int d^4x\,\mathcal{H}_I(x)}\right)\,|\,\mathbf{p}_0\,\rangle$$

Where,

$$\mathcal{H}_I \,=\, \mu\,\Phi\,\phi^2$$

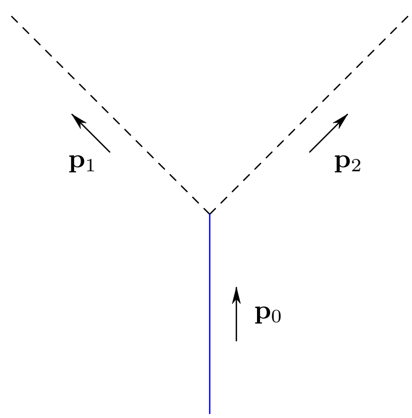
In what follows $p_0 = (M, \mathbf{0})$. For this choice of p_0 the decay rate is given by:

$$d\Gamma \,=\, \frac{1}{2}\,(2M)^{-1}\left(\frac{d^3p_1}{(2\pi)^3}\,\frac{1}{2\,E_{\mathbf{p}_1}}\right)\left(\frac{d^3p_2}{(2\pi)^3}\,\frac{1}{2\,E_{\mathbf{p}_2}}\right)\,|\mathcal{M}|^2\,(2\pi)^4\,\delta^4\,(\,p_0 - (p_1 + p_2))$$

Here the factor of $\frac{1}{2}$ comes from the Bose symmetry of the outgoing ϕ particles. Working to first order in μ we consider the following amplitude:

$$(2\pi)^4\,\delta^4\,(\,p_0 - (p_1 + p_2))\; i\,\mathcal{M} \,=\, -i\,\mu\int d^4x\,\langle\,\mathbf{p}_1\,\mathbf{p}_2\,|\,\Phi(x)\,\phi(x)\,\phi(x)\,|\,\mathbf{p}_0\,\rangle$$

This corresponds to the following Feynman diagram:



Defining $E_{\mathbf{p}}(\mu) \,=\, \sqrt{\mathbf{p}^2 + \mu^2}$, we use the following expansions for the fields:

$$\Phi(x) \,=\, \int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\,E_{\mathbf{p}}(M)}}\,\left(a_{\mathbf{p}}\,e^{-ip\cdot x} \,+\, a^\dagger_{\mathbf{p}}\,e^{ip\cdot x}\right)\Big|_{p^0=E_{\mathbf{p}}(M)}$$

And,

$$\phi(x) \,=\, \int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\,E_{\mathbf{p}}(m)}}\,\left(b_{\mathbf{p}}\,e^{-ip\cdot x} \,+\, b^\dagger_{\mathbf{p}}\,e^{ip\cdot x}\right)\Big|_{p^0=E_{\mathbf{p}}(m)}$$

We have the states:

$$|\,\mathbf{p}_0\,\rangle \,=\, \sqrt{2\,E_{\mathbf{p}_0}(M)}\;a^\dagger_{\mathbf{p}_0}\,|\,0\,\rangle$$

And,

$$|\,\mathbf{p}_1\,\mathbf{p}_2\,\rangle \;=\; \sqrt{2\,E_{\mathbf{p}_1}(m)}\,\sqrt{2\,E_{\mathbf{p}_2}(m)}\;b^\dagger_{\mathbf{p}_1}\,b^\dagger_{\mathbf{p}_2}\;|\,0\,\rangle$$

From,

$$[\Phi(x), a^\dagger_{\mathbf{p}}] \;=\; \frac{e^{-ip\cdot x}}{\sqrt{2\,E_{\mathbf{p}}(M)}} \qquad \text{and} \qquad [\phi(x), b_{\mathbf{p}}] \;=\; \frac{e^{ip\cdot x}}{\sqrt{2\,E_{\mathbf{p}}(m)}}$$

We find:

$$\langle\,\mathbf{p}_1\,\mathbf{p}_2\,|\;\Phi(x)\,\phi(x)\,\phi(x)\;|\,\mathbf{p}_0\,\rangle \;=\; 2\,e^{-i\,(\,p_0-(p_1+p_2))}$$

Thus $i\mathcal{M} = -2i\mu$ and

$$\Gamma \;=\; \frac{\mu^2}{M}\int\frac{d^3p_1}{(2\pi)^3}\,\frac{1}{2\,E_{\mathbf{p}_1}}\int\frac{d^3p_2}{(2\pi)^3}\,\frac{1}{2\,E_{\mathbf{p}_2}}\;(2\pi)^4\;\delta^4\;(\,p_0-(p_1+p_2))$$

Where here and below $E_{\mathbf{p}} = E_{\mathbf{p}}(m)$. Making use of $p_0 = (M, \mathbf{0})$ and

$$\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2\,E_{\mathbf{p}}} \;=\; \int\frac{d^4p}{(2\pi)^4}\,(2\pi)\,\delta(p^2-m^2)\,\theta(p^0)$$

We find:

$$\Gamma \;=\; \frac{\mu^2}{M}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2\,E_{\mathbf{p}}}\,(2\pi)\,\delta((p_0-p)^2-m^2)\,\theta(M-E_{\mathbf{p}})$$

Using $(p_0-p)^2 = M^2 - 2ME_{\mathbf{p}} + m^2$, we have:

$$\delta((p_0-p)^2-m^2)\,\theta(M-E_{\mathbf{p}}) \;=\; \delta(M^2-2ME_{\mathbf{p}})\,\theta(M-E_{\mathbf{p}}) \;=\; (2M)^{-1}\,\delta(E_{\mathbf{p}}-M/2)$$

From $d^3p = |\mathbf{p}|\,E_{\mathbf{p}}dE_{\mathbf{p}}d\Omega$:

$$\Gamma \;=\; \frac{1}{4\pi}\,\frac{\mu^2}{M^2}\int_0^\infty dE_{\mathbf{p}}\,\sqrt{E_{\mathbf{p}}^2-m^2}\;\delta(E_{\mathbf{p}}-M/2)$$

Or, finally,

$$\Gamma \;=\; \frac{M}{8\pi}\,\frac{\mu^2}{M^2}\,\sqrt{1-(2m/M)^2}$$

3.)

We consider the action for a massless scalar field in d dimensions:

$$S\left[\phi,\gamma\right] \,=\, \tfrac{1}{2}\int d^dx\,\sqrt{\gamma}\,\gamma^{ab}\,\partial_a\phi\,\partial_b\phi$$

Here we write the action in a form that is manifestly invariant under diffeomorphisms. However we will only consider metrics that may be brought to the Minkowsi form ($\gamma_{ab} = \eta_{ab}$) by a suitable coordinate transformation. We consider the behavior of the action under a scale transformation:

$$S\left[e^{\alpha\sigma}\phi, e^{2\sigma}\gamma\right] \,=\, \tfrac{1}{2}\int d^dx\,\sqrt{\gamma}\,e^{(d-2)\sigma}\,\gamma^{ab}\,e^{2\alpha\sigma}\,\partial_a\phi\,\partial_b\phi \,=\, e^{(2\alpha+(d-2))\sigma}S\left[\phi,\gamma\right]$$

Thus the action is invariant if $\alpha = -(d-2)/2$. This transformation takes a perhaps more familiar form if we compose it with a diffeomorphism which compensates for the transformation of the metric. Such a transformation amounts to a coordinate change under which the action is trivially invariant. That is we consider:

$$S\left[e^{\alpha\sigma}\phi, e^{2\sigma}\gamma\right] \,=\, S\left[e^{\alpha\sigma}\tilde{\phi}, e^{2\sigma}\tilde{\gamma}\right]$$

Where,

$$\tilde{\phi}(\tilde{x}) \,=\, \phi(x) \qquad \text{and} \qquad \tilde{\gamma}_{ab}(\tilde{x}) \,=\, \frac{\partial x^c}{\partial\tilde{x}^a}\,\frac{\partial x^d}{\partial\tilde{x}^b}\,\gamma_{cd}(x)$$

Choosing the x coordinate system such that $\gamma_{ab} = \eta_{ab}$ and taking $\tilde{x} = e^\sigma x$ we find:

$$e^{\alpha\sigma}\tilde{\phi}(x) \,=\, e^{\alpha\sigma}\phi(e^{-\sigma}x) \qquad \text{and} \qquad \tilde{\gamma}_{ab}(x) \,=\, \gamma_{ab}(e^{-\sigma}x) \,=\, \eta_{ab}$$

Thus writing the action as,

$$S\left[\phi\right] \,\equiv\, S\left[\phi,\eta\right] \,=\, \tfrac{1}{2}\int d^dx\,\eta^{ab}\,\partial_a\phi\,\partial_b\phi$$

If we choose $\alpha = (d-2)/2$, we find that:

$$S\left[\phi\right] \,=\, S\left[\phi'_\sigma\right] \qquad \text{where} \qquad \phi'_\sigma(x) \,=\, e^{-\sigma(d-2)/2}\phi(e^{-\sigma}x)$$

Note that this is not in general equivalent to the application of a coordinate transformation on ϕ, leaving the metric alone, as would be true for a Poincare transformation. However for $d = 2$ the transformation is equivalent to a coordinate transformation applied to ϕ. In fact, for $d = 2$, $S\left[\phi,\gamma\right]$ is invariant under the larger group of conformal transformations for which $\sigma = \Lambda(x)$. This is also true of the action for the electromagnetic field in $d = 4$. These considerations have a consequence for the form of the stress tensor. Consider an infinitesimal variation of the action under a conformal transformation:

$$S\left[e^{\alpha\Lambda}\phi, e^{2\Lambda}\gamma\right] \,-\, S\left[\phi,\gamma\right] \,=\, \int d^dx\,\left[\alpha\,\frac{\delta S}{\delta\phi}\,\phi \,+\, 2\,\frac{\delta S}{\delta\gamma_{ab}}\,\gamma_{ab}\right]\,\Lambda(x)$$

If the action is invariant under a conformal transformation, and thus under a scale transformation, then if $\alpha = 0$ we find:

$$2\,\frac{\delta S}{\delta\gamma_{ab}}\,\gamma_{ab} \,\equiv\, \sqrt{\gamma}\,T^{ab}\,\gamma_{ab} \,=\, 0$$

Thus the stress tensor is traceless for conformally invariant actions when the fields transform trivially. This is true of the massless scalar field in $d = 2$ since, in Minkowski space:

$$T_{ab} \,=\, \partial_a\phi\,\partial_b\phi \,-\, \tfrac{1}{2}\,\eta_{ab}\,\partial_c\phi\,\partial^c\phi$$

It is also true of the electromagnetic field in $d = 4$ where:

$$T_{\alpha\beta} \,=\, F_{\alpha\mu}F_{\nu\beta}\,\eta^{\mu\nu} \,+\, \tfrac{1}{4}\,\eta_{\alpha\beta}\,F_{\mu\nu}F^{\mu\nu}$$

It is important to note that nowhere above is mention made of the equations of motion. Current conservation is a consequence of the equations of motion but symmetries do not involve them. It is a nontrivial statement in a quantum field theory to say that a current associated with a symmetry of the action has a divergence with a vanishing expectation value. Sometimes an anomaly arises which causes the expectation value to be non-zero. Such a violation of the above conformal symmetry arises for the bosonic string and avoiding it fixes the dimension of a flat spacetime to be 26.

(a)

We now consider the group of the Poincare transformations extended by dilations (scale transformations). We consider the action of the Lie algebra of this group acting on functions which have zero scaling dimension (α above). Thus for such a function $f(x)$ we have:

$$\tilde{f}(\tilde{x}) \,=\, f(x) \qquad \text{or} \qquad \tilde{f} \,=\, \exp\left[\, i\, \lambda^\alpha\, G_\alpha \,\right]\, f$$

Here G^α is an element of the Lie algebra with λ_α parameterizing the group manifold. The respective generators are defined as follows:

Translations: $\tilde{x}^\mu = x^\mu + a^\mu$

$$\lambda^\alpha\, G_\alpha \,=\, a^b\, P_b \qquad \text{where} \qquad P_b \,=\, i\, \partial_b$$

Lorentz Transformations: $\tilde{x}^\mu = \exp\left[-i\, \tfrac{1}{2}\, \omega^{ab}\, L_{ab}\right]\, x^\mu$

$$\lambda^\alpha\, G_\alpha \,=\, \tfrac{1}{2}\, \omega^{ab}\, L_{ab} \qquad \text{where} \qquad L_{ab} \,=\, i\, \left(\, x_a\, \partial_b \,-\, x_b\, \partial_a \,\right)$$

Dilations: $\tilde{x}^\mu = e^\sigma\, x^\mu$

$$\lambda^\alpha\, G_\alpha \,=\, \sigma\, D \qquad \text{where} \qquad D \,=\, i\, x^b\, \partial_b$$

We now compute the algebra. We know that (see P&S):

$$[P_\mu, P_\nu] \,=\, 0 \qquad\qquad [L_{\mu\nu}, P_\alpha] \,=\, i\,(\, \eta_{\nu\alpha} P_\mu - \eta_{\mu\alpha} P_\nu \,)$$

And,

$$[L_{\mu\nu}, L_{\alpha\beta}] \,=\, i\,(\, \eta_{\nu\alpha} L_{\mu\beta} - \eta_{\mu\alpha} L_{\nu\beta} - \eta_{\nu\beta} L_{\mu\alpha} + \eta_{\mu\beta} L_{\nu\alpha} \,)$$

We compute:

$$[P_\mu, D] \,=\, -[\partial_\mu, x^\alpha \partial_\alpha] \,=\, -\partial_\mu \,=\, i\, P_\mu$$

Also,

$$[L_{\mu\nu}, D] \,=\, -[x_\mu\, \partial_\nu \,-\, x_\nu\, \partial_\mu, x^\alpha \partial_\alpha]$$

Now,

$$[x_\mu\, \partial_\nu, x^\alpha \partial_\alpha] \,=\, x_\mu [\partial_\nu, x^\alpha] \partial_\alpha \,-\, x^\alpha [\partial_\alpha, x_\mu] \partial_\nu \,=\, 0$$

Thus,

$$[L_{\mu\nu}, D] \,=\, 0$$

We now derive the Noether current associated with the scaling symmetry. We return to the scalar field Lagrangian:

$$S\,[\phi] \,=\, \int d^d x\, \mathcal{L}\,[\phi] \,=\, \tfrac{1}{2} \int d^d x\, \eta^{ab}\, \partial_a \phi\, \partial_b \phi$$

Under the scaling transformation,

$$\phi'_\sigma(x) \,=\, e^{-\sigma(d-2)/2}\, \phi(e^{-\sigma} x)$$

We find:

$$\mathcal{L}'_\sigma\,[\phi]\,(x) \;\equiv\; \mathcal{L}\,[\phi'_\sigma]\,(x) \;=\; e^{-\sigma d}\,\mathcal{L}\,[\phi]\,(e^{-\sigma}x)$$

Thus $\mathcal{L}$ transforms as a density. We could have found the scaling dimension of ϕ by imposing this property but it is perhaps more satisfying to have this property manifest and explicitly introduce the scaling of the metric.

We derive the Noether currents as in P&S:

$$\delta\mathcal{L} \;=\; \partial_a\left(\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\,\delta\phi\right)$$

Now,

$$\delta\mathcal{L} \;=\; \mathcal{L}'_\epsilon - \mathcal{L} \;=\; -\epsilon\,(d\,+\,x^a\partial_a)\,\mathcal{L} \;=\; -\epsilon\,\partial_a\,(x^a\mathcal{L})$$

And,

$$\delta\phi \;=\; \phi'_\epsilon - \phi \;=\; -\epsilon\,[(d-2)/2\,+\,x^a\partial_a]\,\phi$$

Thus,

$$\partial_a\,(x^a\mathcal{L}) \;=\; \partial_a\left(\partial^a\phi\,\left[(d-2)/2\,+\,x^b\partial_b\right]\phi\right)$$

From,

$$T^{ab} \;=\; \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\,\partial^b\phi \;-\; \eta^{ab}\mathcal{L}$$

We find,

$$x^a\mathcal{L} \;=\; \partial^a\phi\,x^b\,\partial_b\phi \;-\; T^{ab}x_b$$

Thus we have the Noether current:

$$J^a_D \;=\; T^{ab}\,x_b \;+\; (d/2-1)\,\phi\,\partial^a\phi$$

Note the second term arises since the stress tensor is not in general traceless.

We now consider the Noether charge $\hat{D}$. Setting $\alpha = (1-d/2)$:

$$\hat{D} \;=\; \int d^{(d-1)}x\,\left(T^{0a}\,x_a - \alpha\,\phi\,\partial^0\phi\right) \;=\; t\,\hat{H} \;+\; \int d^{(d-1)}x\,\left(\mathcal{P}_j\,x^j - \alpha\,\phi\,\pi\right)$$

Here we have defined:

$$T^{0a} \;=\; \mathcal{P}^a \qquad\text{and}\qquad \hat{P}^a \;=\; \int d^{(d-1)}x\,\mathcal{P}^a$$

Note that we use G to denote a generic generator of the action of a symmetry on classical fields and $\hat{G}$ to denote the corresponding Noether charge. The action of the Poincare and scaling symmetries on a Heisenberg operator $O[\phi,\pi](x)$ is given by:

$$O_\lambda \;=\; \exp\left[i\lambda^\alpha\hat{G}_\alpha\right]\,O\,\exp\left[-i\lambda^\alpha\hat{G}_\alpha\right]$$

Note that we have included an explicit coordinate dependence in O since this appears in the Lorentz and dilation charges. This leads to the relation:

$$i\,\frac{\partial O_\lambda}{\partial\lambda^\alpha} \;=\; [O_\lambda,\hat{G}_\alpha]$$

There is the possibilty of some confusion here since the fields can explicitly depend on the coordinates x which are closely related to the translation symmetry parameters.

For example, the Noether charges themselves are independent of time by definition but do not all commute with the Hamiltonian. Those that do not, the dilation and boost generators, explicitly depend on time. For the spacetime derivative of an operator we compute:

$$i\,\partial_a O(x) \;=\; i\,\frac{\partial}{\partial x^a}\,O(x) \;+\; [O(x),\hat{P}_a]$$

The commutator serves to implement the derivative on the field variables. This notation is admittedly somewhat somewhat ambiguous but an example involving the dilation charge should add some clarity:

$$i\,\partial_0\,\hat{D}\,\equiv\,i\,\frac{d}{dt}\,\hat{D}\,=\,i\,\frac{\partial}{\partial t}\,\hat{D}\,+\,[\hat{D},H]$$

Below we will show that, as expected, $[\hat{D},\hat{H}]=-i\hat{H}$. Thus,

$$i\,\frac{d}{dt}\,\hat{D}\,=\,i\,\frac{\partial}{\partial t}\,\hat{D}\,-\,i\,\hat{H}\,=\,0$$

We now compute $[\hat{D},\phi(x)]$. Using $\mathcal{P}_j=\pi\,\partial_j\phi$, we find:

$$\begin{aligned}[\hat{D},\phi(x)] &= t\,[\hat{H},\phi(x)]\,+\,\int d^3y\,[\pi(y),\phi(x)]\left(y^j\,\partial_j\phi-\alpha\,\phi\right)\\ &= -i\,(-\alpha\,+\,x^a\partial_a)\,\phi(x)\end{aligned}$$

Thus if we define,

$$\phi_\sigma\,=\,\exp\left[i\sigma\hat{D}\right]\,\phi\,\exp\left[-i\sigma\hat{D}\right]$$

We have:

$$i\,\frac{\partial\phi_\sigma}{\partial\sigma}\,=\,[\phi_\sigma,\hat{D}]\,=\,D\,\phi$$

Where for scaling dimension α :

$$D\,=\,-i\,\alpha\,+\,i\,x^a\,\partial_a$$

Thus,

$$\phi_\sigma\,=\,\exp\left[-i\sigma D\right]\phi$$

When we were considering the dilation symmetry acting on classical fields above we found:

$$\phi_\sigma\,=\,\exp\left[\,i\sigma D\right]\phi \qquad \text{(classical)}$$

It is important to understand how this minus sign arises. The issue is briefly discussed on pages 59-60 of P&S. Essentially it is because a classical field is the expectation value of the corresponding quantum field. Consider the expectation value $V(x)$ of the operator $O(x)$ in the Heisenberg state $|\Psi\rangle$:

$$V(x)\,=\,\langle\Psi|\,O(x)\,|\Psi\rangle\,=\,\langle\Psi_\lambda|\,O_\lambda(x)\,|\Psi_\lambda\rangle$$

Where, for some Noether charge $\hat{G}$:

$$O_\lambda\,=\,\exp\left[i\lambda\hat{G}\right]\,O\,\exp\left[-i\lambda\hat{G}\right] \qquad \text{and} \qquad |\Psi_\lambda\rangle\,=\,\exp\left[i\lambda\hat{G}\right]|\Psi\rangle$$

Now, we define:

$$V_\lambda(x)\,=\,\exp\left[\,i\lambda\,G\right]V(x)\,\equiv\,\langle\Psi_\lambda|\,O(x)\,|\Psi_\lambda\rangle$$

This serves as a definition of G in terms of $\hat{G}$. Thus,

$$V_\lambda(x)\,=\,\langle\Psi|\,O_{-\lambda}(x)\,|\Psi\rangle\,=\,\exp\left[\,i\lambda\,G\right]\langle\Psi|\,O(x)\,|\Psi\rangle$$

Thus we find:

$$O_\lambda(x)\,=\,\exp\left[-i\lambda\,G\right]O(x)$$

We now compute $[\hat{D},\hat{P}_a]$:

$$[\hat{D},\hat{P}_a]\,=\,\int d^{(d-1)}x\left([\mathcal{P}_j(x),\hat{P}_a]\,x^j\,-\,\alpha\,[\phi(x)\,\pi(x),\hat{P}_a]\right)$$

Where we have used $[\hat{H}, \hat{P}_a] = 0$. Now,

$$\hat{P}_0 \,=\, \hat{H} \,=\, \int d^{(d-1)}x \, \tfrac{1}{2} \left(\pi^2 \,+\, (\nabla\phi)^2 \right) \qquad \text{and} \qquad \hat{P}_j \,=\, \int d^{(d-1)}x \, \pi \, \partial_j \phi$$

For operators without explicit time dependence:

$$i \, \partial_a O(x) \,=\, [O(x), \hat{P}_a]$$

Thus,

$$[\hat{D}, \hat{P}_k] \,=\, i \int d^{(d-1)}x \left[\left(\partial_k \mathcal{P}_j \right) x^j \,-\, \alpha \, \partial_k \left(\phi \, \pi \right) \right] \,=\, -i \, \hat{P}_k$$

Where we have integrated by parts. Also,

$$[\hat{D}, \hat{H}] \,=\, i \int d^{(d-1)}x \left[\left(\nabla^2 \phi \, \partial_j \phi \,+\, \pi \, \partial_j \pi \right) x^j \,-\, \alpha \, \left(\phi \, \nabla^2 \phi \,+\, \pi \, \pi \right) \right]$$

Where we have used $\dot{\phi} = \pi$ and $\dot{\pi} = \ddot{\phi} = \nabla^2 \phi$. Integrating by parts,

$$\begin{aligned} & i \int d^{(d-1)}x \left(\nabla^2 \phi \, \partial_j \phi \,+\, \pi \, \partial_j \pi \right) x^j \\ & \qquad = \; i \int d^{(d-1)}x \, \left[\tfrac{1}{2} \, \partial_j \left(\pi^2 - (\nabla\phi)^2 \right) \,-\, \partial_k \left(\partial^k \phi \, \partial_j \phi \right) \right] x^j \\ & \qquad = \; -i \int d^{(d-1)}x \, \left[\tfrac{1}{2} \, (d-1) \left(\pi^2 - (\nabla\phi)^2 \right) \,+\, (\nabla\phi)^2 \right] \end{aligned}$$

And,

$$-i \, \alpha \int d^{(d-1)}x \left(\phi \, \nabla^2 \phi \,+\, \pi \, \pi \right) \,=\, -i \, \alpha \int d^{(d-1)}x \left(\pi^2 - (\nabla\phi)^2 \right)$$

Thus, since $\alpha = (1 - d/2)$, we find:

$$[\hat{D}, \hat{H}] \,=\, -i \int d^{(d-1)}x \, \tfrac{1}{2} \left(\pi^2 \,+\, (\nabla\phi)^2 \right) \,=\, -i \, \hat{H}$$

Thus, as expected:

$$[\hat{D}, \hat{P}_a] \,=\, -i \, \hat{P}_a$$

(b)

We consider the action for a one-dimensional lattice of particles interacting through nearest neighbor harmonic potentials:

$$S[q] \;=\; \int dt \sum_i \tfrac{1}{2}\left(m\,(\dot{q}_i)^2 - k\,(q_{i+1}-q_i)^2 \right)$$

We now define a continuum field $\phi(x)$ such that on a lattice $x_{i+1} = x_i + a$ we have:

$$\phi(x_i) \;=\; \sqrt{m/a}\,q_i \qquad \text{and} \qquad k = a^{-2}\,m\,c_s^2$$

This leads to:

$$S[q] \;=\; \tfrac{1}{2}\int dt\,a \sum_i \left[(\dot{\phi}(x_i))^2 \;-\; c_s^2\,a^{-2}\,(\phi(x_i+a)-\phi(x_i))^2 \right]$$

Taking a to be small we write:

$$S[\phi] \;=\; \tfrac{1}{2}\int dt\,dx\;\left[(\partial_t\phi)^2 \;-\; c_s^2\,a^{-2}\,(\phi(x+a)-\phi(x))^2 \right]$$

Strictly speaking the limit necessary to define the integral will lead to a free theory. We will simply take $S[\phi]$ to define the physical system. Expanding $\phi(x+a)$ and retaining terms quadratic in a we find:

$$a^{-2}\,(\phi(x+a)-\phi(x))^2 \;=\; (\partial_x\phi)^2 \;+\; a\,(\partial_x\phi)\,(\partial_x^2\phi) \;-\; \tfrac{1}{12}\,a^2\,(\partial_x^2\phi)^2$$

We neglect the second term since it is a total derivative. This leads to,

$$S[\phi] \;=\; S_0[\phi] \;+\; S_I[\phi] \;=\; S_0[\phi] \;+\; \frac{a^2 c_s^2}{24}\int dt\,dx\,(\partial_x^2\phi)^2$$

Where $S_0[\phi]$ is the conformally invariant free Lagrangian:

$$S_0[\phi] \;=\; \tfrac{1}{2}\int dt\,dx\;\left((\partial_t\phi)^2 - c_s^2\,(\partial_x\phi)^2 \right)$$

The Lagrangian is invariant under the dilation:

$$\phi'_\sigma(t,x) \;=\; \phi(\,e^{-\sigma}t\,,\,e^{-\sigma}x\,)$$

The associated free Hamiltonian scales as:

$$H_0[\phi'_\sigma] \;=\; e^{-\sigma}\,H_0[\phi]$$

Thus the dilation maps higher energy solutions to lower energy solutions. The action of the dilation on the interaction is:

$$S_I[\phi'_\sigma] \;=\; e^{-2\sigma}\,S_I[\phi]$$

Thus the interaction becomes irrelevant at low energies.

For a plane wave solution of momentum p we find:

$$H_I/H_0 \;\simeq\; p^2\,a^2/12$$

Thus when 10% of the energy is in the interaction $p \simeq \sqrt{6}/(a\,\sqrt{5})$.