QFT 1 : Problem Set 6

1.) Peskin & Schroeder 4.3

We consider N real scalar fields interacting according to the Hamiltonian:

$$H \;=\; \int d^3x \,\left(\tfrac{1}{2}\,\mathbf{\Pi}\cdot\mathbf{\Pi} \,+\, \tfrac{1}{2}\,\nabla\mathbf{\Phi}\cdot\nabla\mathbf{\Phi} \,+\, V(\mathbf{\Phi}\cdot\mathbf{\Phi}) \right)$$

Where $\mathbf{\Phi} = (\Phi_1 \ldots \Phi_N)$, $\mathbf{\Pi}$ is the conjugate momentum and

$$V(\mathbf{\Phi}\cdot\mathbf{\Phi}) \;=\; \tfrac{1}{2}\,m^2\,\mathbf{\Phi}\cdot\mathbf{\Phi} \,+\, \frac{\lambda}{4}\,(\mathbf{\Phi}\cdot\mathbf{\Phi})^2$$

(a)

For $\lambda = 0$ we have N decoupled Klein-Gordon fields of equal mass. This leads directly to the following definition of the field in the interaction picture:

$$\Phi_j(x) \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,\left(a_{j\mathbf{p}}\,e^{-ip\cdot x} \,+\, a^\dagger_{j\mathbf{p}}\,e^{ip\cdot x} \right)$$

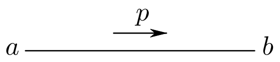
Where $E_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2$. The canonical commutation relations lead to:

$$[a_{i\mathbf{p}}, a^\dagger_{j\mathbf{k}}] \;=\; \delta_{ij}\,(2\pi)^3\,\delta(\mathbf{p}-\mathbf{k}) \qquad\qquad [a_{i\mathbf{p}}, a_{j\mathbf{k}}] \;=\; [a^\dagger_{i\mathbf{p}}, a^\dagger_{j\mathbf{k}}] \;=\; 0$$

This leads to the spacetime propagator:

$$\contraction{}{\Phi}{_i(x)\,}{\Phi}\Phi_i(x)\,\Phi_j(y) \;=\; \langle 0|\,T\,(\Phi_i(x)\,\Phi_j(y))\,|0\rangle \;=\; \delta_{ij}\,D_F(x-y)$$

This leads to the momentum-space Feynman rule:



$$\Rightarrow \qquad \frac{i\,\delta_{ab}}{p^2 - m^2 + i\epsilon}$$

We compute the following amplitude to $\mathcal{O}(\lambda)$:

$$i\,\mathcal{M}(p'_k, q'_l\,|\,p_i, q_j)\,(2\pi)^4\,\delta(p_i + p_j - p_k - p_l) \;=\; \langle p'_k, q'_l|\,i\,\mathcal{T}\,|p_i, q_j\rangle$$

Where,

$$S = 1 \,+\, i\,\mathcal{T} \;=\; T\left(e^{-i\int d^4x\,\mathcal{H}_I(x)} \right)$$

And,

$$|p_i, q_j\rangle \;=\; \sqrt{2E_{\mathbf{p}_i}}\,\sqrt{2E_{\mathbf{q}_j}}\;a^\dagger_{i\mathbf{p}_i}\,a^\dagger_{j\mathbf{q}_j}\;|0\rangle$$

Of course the interaction Hamiltonian is:

$$\mathcal{H}_I(x) \;=\; \frac{\lambda}{4}\,(\mathbf{\Phi}\cdot\mathbf{\Phi})^2$$

To $\mathcal{O}(\lambda)$:

$$\langle p'_k, q'_l|\,i\,\mathcal{T}\,|p_i, q_j\rangle \;=\; -i\,\frac{\lambda}{4}\int d^4x\;\langle p'_k, q'_l|\,\Phi_{ax}\Phi_{ax}\Phi_{bx}\Phi_{bx}\,|p_i, q_j\rangle$$

Here summation over a and b is implied and we have defined $\Phi_{ax} \equiv \Phi_a(x)$.

Ignoring vacuum bubbles and taking account of the symmetry under exchange of a and b as well as under exchange of fields within both of the pairings we find:

$$\begin{aligned}
\langle p'_k, q'_l|\, \Phi_{ax}\Phi_{ax}\Phi_{bx}\Phi_{bx}\,|p_i, q_j\rangle \;&=\; 8\;\wick{\langle \c1 p'_k, \c2 q'_l|\, \c1 \Phi_{ax} \c2 \Phi_{ax} \c3 \Phi_{bx} \c4 \Phi_{bx}\,| \c3 p_i, \c4 q_j\rangle} \\
&+\; 8\;\wick{\langle \c1 p'_k, \c2 q'_l|\, \c1 \Phi_{ax} \c3 \Phi_{ax} \c2 \Phi_{bx} \c4 \Phi_{bx}\,| \c3 p_i, \c4 q_j\rangle} \\
&+\; 8\;\wick{\langle \c1 p'_k, \c2 q'_l|\, \c1 \Phi_{ax} \c4 \Phi_{ax} \c2 \Phi_{bx} \c3 \Phi_{bx}\,| \c3 p_i, \c4 q_j\rangle}
\end{aligned}$$

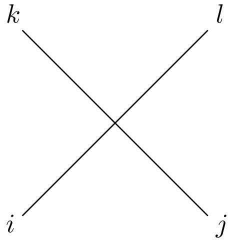
Now,

$$\wick{\c1 \Phi_a(x)\,|\c1 p_b\rangle} \;=\; \delta_{ab}\, e^{-ip\cdot x}\;|0\rangle \qquad \text{and} \qquad \wick{\langle \c1 p_b|\, \c1 \Phi_a(x)} \;=\; \langle 0|\; e^{ip\cdot x}\,\delta_{ab}$$

Thus,

$$i\,\mathcal{M}(k,l\,|\,i,j) \;\equiv\; i\,\mathcal{M}(p'_k, q'_l\,|\,p_i, q_j) \;=\; -2i\,\lambda\;(\,\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\,)$$

This leads to the vertex Feynman rule:



$$\Rightarrow \qquad -2i\,\lambda\;(\,\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\,)$$

We now consider scattering processes in the center of mass frame. Since the masses are identical we may use the formula (eq. 4.85) in P&S for the differential cross section:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} \;=\; \frac{|\mathcal{M}|^2}{64\pi^2 E_{CM}^2} \qquad \text{where} \qquad E_{CM}^2 \;=\; (p_i + q_j)^2$$

For $\Phi_1\Phi_2 \to \Phi_1\Phi_2$ scattering, $i\mathcal{M} = -2i\lambda$. Thus,

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} \;=\; \frac{\lambda^2}{16\pi^2 E_{CM}^2} \qquad \text{for} \qquad \Phi_1\Phi_2 \to \Phi_1\Phi_2$$

For $\Phi_1\Phi_1 \to \Phi_2\Phi_2$ scattering, $i\mathcal{M} = -2i\lambda$. We also divide by a Bose factor of 2. Thus,

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} \;=\; \frac{\lambda^2}{32\pi^2 E_{CM}^2} \qquad \text{for} \qquad \Phi_1\Phi_1 \to \Phi_2\Phi_2$$

For $\Phi_1\Phi_1 \to \Phi_1\Phi_1$ scattering, $i\mathcal{M} = -6i\lambda$. We again divide by a Bose factor of 2. Thus,

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} \;=\; \frac{9\,\lambda^2}{32\pi^2 E_{CM}^2} \qquad \text{for} \qquad \Phi_1\Phi_1 \to \Phi_1\Phi_1$$

(b)

We now consider the potential:

$$V(\mathbf{\Phi}\cdot\mathbf{\Phi}) \;=\; -\tfrac{1}{2}\,\mu^2\,\mathbf{\Phi}\cdot\mathbf{\Phi} \;+\; \frac{\lambda}{4}\,(\mathbf{\Phi}\cdot\mathbf{\Phi})^2$$

We compute the minimum of the potential:

$$\frac{\partial V}{\partial \Phi_j} = -\mu^2\,\Phi_j + \lambda\,\Phi_j\,\mathbf{\Phi}\cdot\mathbf{\Phi}$$

Thus there is a minimum at:

$$\upsilon^2 = \mathbf{\Phi}\cdot\mathbf{\Phi} = \mu^2/\lambda$$

We make the field redefinitions:

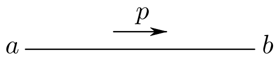
$$\Phi_j(x) = \pi_j(x) \qquad i = 1\ldots N-1 \qquad\qquad \text{and} \qquad\qquad \Phi_N(x) = \upsilon + \sigma(x)$$

Ignoring constant terms, this leads to the potential:

$$V(\sigma, \boldsymbol{\pi}\cdot\boldsymbol{\pi}) = \tfrac{1}{2}\,m_\sigma\,\sigma^2 + \frac{\lambda}{4}\left(\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)^2 + \mu\sqrt{\lambda}\,\sigma\left(\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)$$

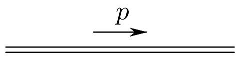
Where $m_\sigma = 2\mu^2$. Since the field displacement υ does not affect the kinetic terms, we may read off the Feynman rules for the propagators and the quartic interactions from the results of part (a) above.

The π_j fields are massless:



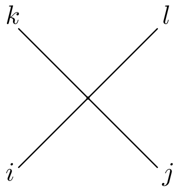
$$\Rightarrow \qquad \frac{i\,\delta_{ab}}{p^2 + i\epsilon}$$

There is also the single massive field σ:



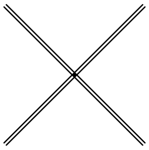
$$\Rightarrow \qquad \frac{i}{p^2 + m_\sigma^2 + i\epsilon}$$

The four π vertex is as found above:



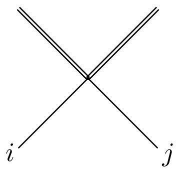
$$\Rightarrow \qquad -2i\,\lambda\,(\,\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\,)$$

The four σ vertex is:



$$\Rightarrow \qquad -6i\,\lambda$$

The two σ two π vertex is:



$$\Rightarrow \qquad -2i\,\lambda\,\delta_{ij}$$

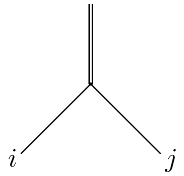
For the one σ two π vertex we compute:

$$i\mathcal{M}\left(\sigma(q)|\,\pi_i(p)\pi_j(k)\right)\,(2\pi)^4\,\delta^4(p+k-q)\;=\;\langle\sigma(q)|\,i\,\mathcal{T}\,|\pi_i(p)\pi_j(k)\rangle$$

Where,

$$\begin{aligned}
\langle\sigma(q)|\,i\,\mathcal{T}\,|\pi_i(p)\pi_j(k)\rangle &= -i\,\mu\,\sqrt{\lambda}\int d^4x\;\langle\sigma(q)|\,\sigma_x\pi_{ax}\pi_{ax}\,|\pi_i(p)\pi_j(k)\rangle\\
&= -2i\,\mu\,\sqrt{\lambda}\int d^4x\;\wick{\langle\c1\sigma(q)|\,\c1\sigma_x\c2\pi_{ax}\c3\pi_{ax}\,|\c2\pi_i(p)\c3\pi_j(k)}\rangle\\
&= -2i\,\mu\,\sqrt{\lambda}\,\delta_{ij}\,(2\pi)^4\,\delta^4(p+k-q)
\end{aligned}$$

Thus we find the vertex rule:



$$\Rightarrow\qquad -2i\,\mu\,\sqrt{\lambda}\,\delta_{ij}$$

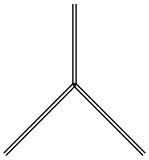
For the three σ vertex we compute:

$$i\mathcal{M}\left(\sigma(q)|\,\sigma(p)\sigma(k)\right)\,(2\pi)^4\,\delta^4(p+k-q)\;=\;\langle\sigma(q)|\,i\,\mathcal{T}\,|\sigma(p)\sigma(k)\rangle$$

Where,

$$\begin{aligned}
\langle\sigma(q)|\,i\,\mathcal{T}\,|\sigma(p)\sigma(k)\rangle &= -i\,\mu\,\sqrt{\lambda}\int d^4x\;\langle\sigma(q)|\,\sigma_x\sigma_x\sigma_x\,|\sigma(p)\sigma(k)\rangle\\
&= -6i\,\mu\,\sqrt{\lambda}\int d^4x\;\wick{\langle\c1\sigma(q)|\,\c1\sigma_x\c2\sigma_x\c3\sigma_x\,|\c2\sigma(p)\c3\sigma(k)}\rangle\\
&= -6i\,\mu\,\sqrt{\lambda}\,(2\pi)^4\,\delta^4(p+k-q)
\end{aligned}$$

Thus we find the vertex rule:

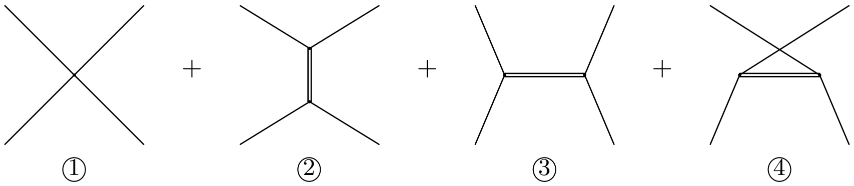


$$\Rightarrow\qquad -6i\,\mu\,\sqrt{\lambda}$$

(c)

We now compute the four π scattering amplitude to $\mathcal{O}(\lambda)$:

$$i\,\mathcal{M}\left(\,\pi_k(p_3)\pi_l(p_4)|\,\pi_i(p_1)\pi_j(p_2)\,\right)\;=$$



Using the Feynman rules derived above:

$$\begin{aligned}
① &= -2i\,\lambda\,(\,\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\,) \\
② &= -2i\,\lambda\,\delta_{ij}\,\delta_{kl}\,\frac{m_\sigma^2}{(p_1+p_2)^2 - m_\sigma^2} \\
③ &= -2i\,\lambda\,\delta_{ik}\,\delta_{jl}\,\frac{m_\sigma^2}{(p_1-p_3)^2 - m_\sigma^2} \\
④ &= -2i\,\lambda\,\delta_{il}\,\delta_{jk}\,\frac{m_\sigma^2}{(p_1-p_4)^2 - m_\sigma^2}
\end{aligned}$$

Here we have used $m_\sigma^2 = 2\,\mu^2$. Thus,

$$i\,\mathcal{M} \;=\; -2i\,\lambda \left(\frac{\delta_{ij}\,\delta_{kl}\,(p_1+p_2)^2}{(p_1+p_2)^2 - m_\sigma^2} + \frac{\delta_{ik}\,\delta_{jl}\,(p_1-p_3)^2}{(p_1-p_3)^2 - m_\sigma^2} + \frac{\delta_{il}\,\delta_{jk}\,(p_1-p_4)^2}{(p_1-p_4)^2 - m_\sigma^2} \right)$$

Since the pions are massless, clearly at threshold ($\mathbf{p}_a = 0$) this amplitude vanishes.

For $N = 2$ we have $\delta_{ij} = 1$ since there is only one species of pion. To $\mathcal{O}(p_a^2)$:

$$\begin{aligned}
i\,\mathcal{M} &= \frac{2i\,\lambda}{m_\sigma^2}\,\big(\,(p_1+p_2)^2 + (p_1-p_3)^2 + (p_1-p_4)^2\,\big) \\
&= \frac{4i\,\lambda}{m_\sigma^2}\,(\,p_1\cdot p_2 - p_1\cdot p_3 - p_1\cdot p_4\,) \\
&= -\frac{4i\,\lambda}{m_\sigma^2}\,(\,p_1\cdot p_1\,) = 0
\end{aligned}$$

Where we have used $p_1 + p_2 = p_3 + p_4$.

(d)

We now add a symmetry breaking term to the Lagrangian:

$$\Delta V \;=\; -\,a\,\Phi_N$$

We express the Lagrangian in terms of the fields:

$$\Phi_j(x) \;=\; \pi_j(x) \qquad i = 1\ldots N-1 \qquad\quad \text{and} \qquad\quad \Phi_N(x) \;=\; \upsilon + \sigma(x) + \delta$$

Here δ is small and we expect a minimum of the potential at $\boldsymbol{\pi} = \sigma = 0$. We derived an expression for the unperturbed potential in terms of $\boldsymbol{\pi}$ and σ in part (c) above. We simply substitute $\sigma(x) \to \sigma(x) + \delta$ into this expression and find the minimum of the potential by varying δ. Thus,

$$\tilde{V}(\sigma, \boldsymbol{\pi}\cdot\boldsymbol{\pi}) \;=\; V(\sigma+\delta, \boldsymbol{\pi}\cdot\boldsymbol{\pi}) \;-\; a\,(\sigma + \delta + \upsilon)$$

Where, as above:

$$V(\sigma, \boldsymbol{\pi}\cdot\boldsymbol{\pi}) \;=\; \mu^2\,\sigma^2 + \frac{\lambda}{4}\,\big(\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi}\big)^2 + \mu\,\sqrt{\lambda}\,\sigma\,\big(\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi}\big)$$

Now, ignoring terms quadratic and higher in δ:

$$\left.\frac{\partial\tilde{V}}{\partial\delta}\right|_{\boldsymbol{\pi}=\sigma=0} \;=\; 2\,\mu^2\,\delta \;-\; a$$

Thus $\delta = a/(2\mu^2)$. Working to $\mathcal{O}(a)$ and ignoring constant terms:

$$\tilde{V}(\sigma, \boldsymbol{\pi}\cdot\boldsymbol{\pi}) \;=\; \tfrac{1}{2}\, m_\sigma^2\, \sigma^2 \;+\; \tfrac{1}{2}\, m_\pi^2\, \boldsymbol{\pi}\cdot\boldsymbol{\pi} \;+\; \frac{\lambda}{4}\,\left(\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)^2 \;+\; \gamma\,\mu\,\sqrt{\lambda}\,\sigma\,\left(\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)$$

Where,

$$m_\sigma^2 \;=\; 2\,\mu^2 \;+\; 3\,\sqrt{\lambda}\,\frac{a}{\mu} \qquad \text{and} \qquad m_\pi^2 \;=\; \sqrt{\lambda}\,\frac{a}{\mu}$$

And,

$$\gamma \;=\; 1 \;+\; \tfrac{1}{2}\,\sqrt{\lambda}\,\frac{a}{\mu^3}$$

The Feynman rules are given by substituting the modified masses into the π and σ propagators and introducing a factor of γ into the cubic vertex expressions.

We now compute the four pion interaction considered above. The first diagram is unmodified; however the last three terms take the form:

$$\begin{aligned}
② &= -2i\,\lambda\,\delta_{ij}\,\delta_{kl}\,\frac{2\,\gamma^2\,\mu^2}{(p_1+p_2)^2 - m_\sigma^2}\\
③ &= -2i\,\lambda\,\delta_{ik}\,\delta_{jl}\,\frac{2\,\gamma^2\,\mu^2}{(p_1-p_3)^2 - m_\sigma^2}\\
④ &= -2i\,\lambda\,\delta_{il}\,\delta_{jk}\,\frac{2\,\gamma^2\,\mu^2}{(p_1-p_4)^2 - m_\sigma^2}
\end{aligned}$$

At threshold ($\mathbf{p}_a = 0$):

$$\left(p_1+p_2\right)^2 \;=\; 4\,m_\pi^2 \qquad \text{and} \qquad \left(p_1-p_3\right)^2 \;=\; \left(p_1-p_4\right)^2 \;=\; 0$$

After a little algebra we find:

$$i\,\mathcal{M} \;=\; i\,(\lambda)^{3/2}\;\frac{a}{\mu^3}\;(\,3\,\delta_{ij}\,\delta_{kl} \;-\; (\,\delta_{ik}\,\delta_{jl} \;+\; \delta_{il}\,\delta_{jk}\,)\,)$$

2.) Peskin & Schroeder 4.4

We consider the quantized Dirac field interacting with a classical potential $A_\mu(x)$:

$$H_I \;=\; \int d^3x\, e\,\bar\psi\gamma^\mu\psi\, A_\mu$$

(a)

We consider the ($\mathcal{O}(e)$) matrix element:

$$\langle \mathbf{p}'s'|\, i\,\mathcal{T}\,|\mathbf{p}s\rangle \;=\; -ie\int d^4x\, A_\mu(x)\;\langle \mathbf{p}'s'|\,\bar\psi(x)\gamma^\mu\psi(x)\,|\mathbf{p}s\rangle$$

Here $S = 1 + i\mathcal{T}$. Now, ignoring vacuum bubbles:

$$\langle \mathbf{p}'s'|\,\bar\psi(x)\gamma^\mu\psi(x)\,|\mathbf{p}\,s\rangle \;=\; \contraction{\langle \mathbf{p}'}{s'}{|\,}{\bar\psi}\contraction{\langle \mathbf{p}'s'|\,\bar\psi(x)\gamma^\mu}{\psi}{(x)\,|}{\mathbf{p}}\langle \mathbf{p}'s'|\,\bar\psi(x)\gamma^\mu\psi(x)\,|\mathbf{p}\,s\rangle$$

Where,

$$\contraction{}{\psi}{(x)\,|}{\mathbf{p}}\psi(x)\,|\mathbf{p}\,s\rangle \;=\; e^{-ip\cdot x}\,u^s(p)\,|0\rangle \qquad\text{and}\qquad \contraction{\langle \mathbf{p}'}{s'}{|\,}{\bar\psi}\langle \mathbf{p}'s'|\,\bar\psi(x) \;=\; \langle 0|\,\bar u^{s'}(p')\,e^{ip'\cdot x}$$

Thus,

$$\langle \mathbf{p}'s'|\,i\,\mathcal{T}\,|\mathbf{p}\,s\rangle \;=\; -ie\,\bar u^{s'}(p')\gamma^\mu u^s(p)\,\tilde A_\mu(p'-p)$$

Where,

$$\tilde A_\mu(q) \;=\; \int d^4x\, A_\mu(x)\, e^{iq\cdot x}$$

(b)

If $A_\mu(x)$ is time independent:

$$\tilde A_\mu(q) \;=\; \int dt\, e^{iq^0t}\int d^3x\, A_\mu(\mathbf{x})\, e^{-i\mathbf{q}\cdot\mathbf{x}} \;=\; (2\pi)\,\delta(q^0)\,\tilde A_\mu(\mathbf{q})$$

Where it is to be understood that $\tilde A_\mu(\mathbf{q})$ is the spatial Fourier transform of $A_\mu(\mathbf{x})$.

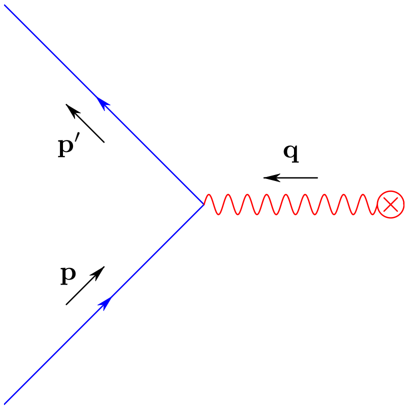
With the following definition of $\mathcal{M}$:

$$\langle \mathbf{p}'s'|\,i\,\mathcal{T}\,|\mathbf{p}\,s\rangle \;=\; i\mathcal{M}\,(p's'\,|\,p\,s)\;(2\pi)\,\delta(E_{\mathbf{p}'}-E_{\mathbf{p}})$$

We find:

$$i\mathcal{M}\,(p's'\,|\,p\,s) \;=\; -ie\,\bar u^{s'}(p')\gamma^\mu u^s(p)\,\tilde A_\mu(\mathbf{p}'-\mathbf{p})$$

Thus, with the usual momentum space Feynman rules for fermions, we find the following vertex rule:



$$\Rightarrow \qquad i\,e\,\gamma^\mu\,\tilde A_\mu(\mathbf{q}) \qquad (q^0=0)$$

We consider the following expression for $d\sigma$:

$$d\sigma \;=\; \frac{d^3p'}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}'}}\int d^2b\,\left|\langle p's'|\,i\,\mathcal{T}\,e^{i\mathbf{b}\cdot\mathbf{P}}\,|\phi\, s\rangle\right|^2$$

Here $e^{i\mathbf{b}\cdot\mathbf{P}}$ is the translation operator perpendicular to the beam. Also,

$$|\phi\, s\rangle \;=\; \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{k}}}}\,\phi(\mathbf{k})\,|k\, s\rangle \qquad \text{where} \qquad \langle\phi\, s|\,\phi\, s\rangle \;=\; \int\frac{d^3k}{(2\pi)^3}\,|\phi(\mathbf{k})|^2 \;=\; 1$$

Note that $\phi(\mathbf{k})$ has support in a highly localized region centered at $\mathbf{p} = p_z\hat{\mathbf{z}}$.

We are essentially considering a mixed incoming state found by integrating over all impact parameters $\mathbf{b}$. We could also include a mixture of incoming spin states but we will defer this for the time being.

Now,

$$\langle p's'|\,i\,\mathcal{T}\,e^{i\mathbf{b}\cdot\mathbf{P}}\,|\phi\, s\rangle \;=\; \int\frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{k}}}}\,\phi(\mathbf{k})\,e^{i\mathbf{b}\cdot\mathbf{k}}\;\langle p's'|\,i\,\mathcal{T}\,|k\, s\rangle$$

Using the definition of $\mathcal{M}$ from above:

$$\begin{aligned} d\sigma \;\;=\;\; & \frac{d^3p'}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}'}}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{k}}}}\int\frac{d^3q}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{q}}}}\,(2\pi)^2\,\delta(E_{\mathbf{p}'}-E_{\mathbf{k}})\,\delta(E_{\mathbf{p}'}-E_{\mathbf{q}}) \\ & \times\,\phi(\mathbf{k})\,\phi^*(\mathbf{q})\,\mathcal{M}\,(p's'\mid k\, s)\;\mathcal{M}^*\,(p's'\mid q\, s)\int d^2b\,e^{i\mathbf{b}\cdot(\mathbf{k}-\mathbf{q})} \end{aligned}$$

Using,

$$\int d^2b\,e^{i\mathbf{b}\cdot(\mathbf{k}-\mathbf{q})} \;=\; (2\pi)^2\,\delta(k_x-q_x)\,\delta(k_y-q_y)$$

And,

$$\delta(E_{\mathbf{p}'}-E_{\mathbf{k}})\,\delta(E_{\mathbf{p}'}-E_{\mathbf{q}}) \;=\; \delta(E_{\mathbf{p}'}-E_{\mathbf{k}})\,\delta(E_{\mathbf{q}}-E_{\mathbf{k}})$$

And defining $\tilde{\mathbf{q}} = q_z\hat{\mathbf{z}} + k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}$, we find:

$$\begin{aligned} d\sigma \;\;=\;\; & \frac{d^3p'}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}'}}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{k}}}}\int\frac{dq_z}{(2\pi)}\,\frac{1}{\sqrt{2E_{\tilde{\mathbf{q}}}}}\,(2\pi)^2\,\delta(E_{\mathbf{p}'}-E_{\mathbf{k}})\,\delta(E_{\tilde{\mathbf{q}}}-E_{\mathbf{k}}) \\ & \times\,\phi(\mathbf{k})\,\phi^*(\tilde{\mathbf{q}})\,\mathcal{M}\,(p's'\mid k\, s)\;\mathcal{M}^*\,(p's'\mid \tilde{q}\, s) \end{aligned}$$

Now,

$$\delta(E_{\tilde{\mathbf{q}}}-E_{\mathbf{k}}) \;=\; \frac{E_{\mathbf{k}}}{|k_z|}\,\left[\,\delta(q_z-k_z)+\delta(q_z+k_z)\,\right]$$

Since $\phi(\mathbf{k})$ is highly localized around $\mathbf{p} = p_z\hat{\mathbf{z}}$, the term produced by $\delta(q_z+k_z)$ vanishes since $\phi(k_z\hat{\mathbf{z}})\,\phi^*(-k_z\hat{\mathbf{z}}) = 0$. Thus,

$$d\sigma \;=\; \frac{d^3p'}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}'}}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2\,|k_z|}\,(2\pi)\,\delta(E_{\mathbf{p}'}-E_{\mathbf{k}})\;|\phi(\mathbf{k})|^2\;|\mathcal{M}\,(p's'\mid k\, s)|^2$$

We now consider $\phi(\mathbf{k})$ to be so highly localized that we may take:

$$|\phi(\mathbf{k})|^2 \;=\; (2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{k})$$

Thus we find:

$$d\sigma \;=\; \frac{d^3p'}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}'}}\,\frac{1}{2\,|p_z|}\,(2\pi)\,\delta(E_{\mathbf{p}'}-E_{\mathbf{p}})\;|\mathcal{M}\,(p's'\mid p\, s)|^2$$

We know integrate over $|\mathbf{p}'|$ to compute $d\sigma/d\Omega$. Since $E_{\mathbf{p}'}^2 = |\mathbf{p}'|^2 + m^2$, we have:

$$d^3p' \;=\; |\mathbf{p}'|^2\;d\,|\mathbf{p}'|\;d\Omega \;=\; |\mathbf{p}'|\;E_{\mathbf{p}'}\,dE_{\mathbf{p}'}\,d\Omega$$

Thus,

$$\frac{d\sigma}{d\Omega} \;=\; \frac{1}{4\,|p_z|}\int_0^\infty dE_{\mathbf{p}'}\,\frac{|\mathbf{p}'|}{(2\pi)^2}\,\delta(E_{\mathbf{p}'}-E_{\mathbf{p}})\;\left|\mathcal{M}\,(p's'\mid p\,s)\right|^2$$

Finally, since the delta function enforces $|\mathbf{p}'| = |\mathbf{p}| = |p_z|$, we find:

$$\frac{d\sigma}{d\Omega} \;=\; \frac{1}{(4\pi)^2}\;\left|\mathcal{M}\,(p's'\mid p\,s)\right|^2$$

(c)

We consider the Coulomb potential:

$$A_\mu(\mathbf{x}) \;=\; \frac{Ze}{4\pi\,|\mathbf{x}|}\,\delta_{\mu 0} \qquad\text{or}\qquad \tilde{A}_\mu(\mathbf{q}) \;=\; \frac{Ze}{|\mathbf{q}|^2}\,\delta_{\mu 0}$$

Thus,

$$i\mathcal{M}\,(p's'\mid p\,s) \;=\; -ie\,\bar{u}^{s'}(p')\gamma^\mu u^s(p)\,\tilde{A}_\mu(\mathbf{p}'-\mathbf{p}) \;=\; \frac{-iZe^2}{|\mathbf{p}'-\mathbf{p}|^2}\,\bar{u}^{s'}(p')\gamma^0 u^s(p)$$

In the non-relativistic limit:

$$u^s(p) \;=\; \sqrt{m}\,\begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix}$$

Thus,

$$\bar{u}^{s'}(p')\gamma^0 u^s(p) \;=\; u^{s'\dagger}(p')\,u^s(p) \;=\; 2\,m\,\delta^{ss'}$$

Since $|\mathbf{p}'| = |\mathbf{p}| = mv$:

$$|\mathbf{p}'-\mathbf{p}|^2 \;=\; |\mathbf{p}|^2\;2\,(1-\cos(\theta)) \;=\; m^2\,v^2\,4\,(\sin(\theta/2))^2$$

Thus,

$$i\mathcal{M}\,(p's'\mid p\,s) \;=\; \frac{-iZe^2}{2mv^2}\;(\sin(\theta/2))^{-2}\;\delta^{ss'}$$

This leads to:

$$\frac{d\sigma}{d\Omega} \;=\; \frac{1}{(4\pi)^2}\;\left|\mathcal{M}\,(p's'\mid p\,s)\right|^2 \;=\; \left(\frac{Ze^2}{4\pi}\right)^2\;\frac{\delta^{ss'}}{4m^2v^4}\;(\sin(\theta/2))^{-4}$$

We now sum over outgoing and average over incoming spins. Using $\alpha = e^2/(4\pi)$ we find:

$$\frac{d\sigma}{d\Omega} \;=\; \frac{Z^2\alpha^2}{4m^2v^4}\;(\sin(\theta/2))^{-4}$$