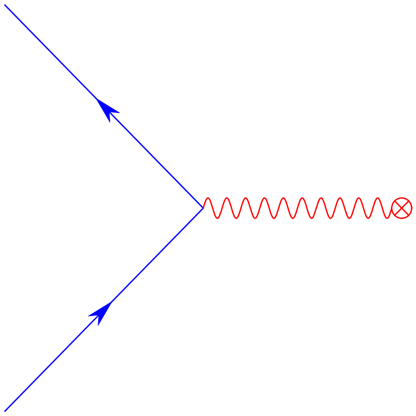
QFT 1 : Problem Set 7

1.) P&S 5.1: Coulomb Scattering

We consider the process:



Where blue lines are electrons and red lines are photons.

We begin with the expression for the differential cross section for scattering from a time-independent and localized potential:

$$d\sigma = \frac{1}{\upsilon_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\left|\mathcal{M}(p_i \to p_f)\right|^2 (2\pi)\delta(E_f - E_i)$$

Where in the case of electron scattering:

$$i\mathcal{M}((p,s) \to (p',s')) = -ie\overline{u}^{s'}(p')\gamma^\mu u^s(p)\widetilde{A}_\mu(\mathbf{q})$$

Where $\widetilde{A}_\mu(\mathbf{q})$ is the Fourier transform of the potential $A_\mu(\mathbf{x})$ evaluated at $\mathbf{q} = \mathbf{p'} - \mathbf{p}$. For the case of the Coulomb potential (Z=1):

$$A_\mu(\mathbf{x}) = \delta_{\mu 0} e/4\pi\left|\mathbf{x}\right|$$

We have for $\widetilde{A}_\mu(\mathbf{q})$:

$$\widetilde{A}_\mu(\mathbf{q}) = \delta_{\mu 0} e/\left|\mathbf{q}\right|^2$$

Taking $d^3p_f = \left|\mathbf{p}_f\right|^2 d\left|\mathbf{p}_f\right| d\Omega$ and making the replacements $p_i \to p = (E, \mathbf{p})$ and $p_f \to p' = (E', \mathbf{p'})$ and also using $\upsilon = \left|\mathbf{p}\right|/E$:

$$\frac{1}{\upsilon_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f} = \frac{1}{2\left|\mathbf{p}\right|}\frac{1}{(2\pi)^3}\left|\mathbf{p'}\right|^2 d\left|\mathbf{p'}\right| d\Omega \frac{1}{2E'}$$

Now since $E'^2 = \left|\mathbf{p'}\right|^2 + m^2$ we have: $\left|\mathbf{p'}\right| d\left|\mathbf{p'}\right| = E'dE'$. Using the fact that $\left|\mathbf{p'}\right| = \left|\mathbf{p}\right|$ due to the delta function:

$$\frac{1}{\upsilon_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f} = \frac{1}{4}\frac{1}{(2\pi)^3}dE'd\Omega$$

Thus,

$$d\sigma = \frac{dE'd\Omega}{(4\pi)^2}\left|\mathcal{M}((p,s) \to (p',s'))\right|^2 \delta(E' - E)$$

We now integrate over the energy of the outgoing particle with the understanding that in what follows $|\mathbf{p}'| = |\mathbf{p}|$. We also sum over outgoing spin states and average over incoming spin states to arrive at:

$$\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi)^2}\frac{1}{2}\sum_{ss'}|\mathcal{M}((p,s)\to(p',s'))|^2$$

where,

$$i\mathcal{M}((p,s)\to(p',s')) = -i\frac{e^2}{|\mathbf{q}|^2}\overline{u}^{s'}(p')\gamma^0 u^s(p)$$

Using $(\overline{\upsilon}\gamma^\nu u)^\dagger = (\overline{u}\gamma^\nu \upsilon)$:

$$\frac{d\sigma}{d\Omega} = (\frac{e^2}{4\pi})^2\frac{1}{|\mathbf{p'}-\mathbf{p}|^4}\frac{1}{2}\sum_{ss'}\overline{u}^s(p)\gamma^0 u^{s'}(p')\overline{u}^{s'}(p')\gamma^0 u^s(p)$$

Now,

$$\sum_{ss'}\overline{u}^s(p)\gamma^0 u^{s'}(p')\overline{u}^{s'}(p')\gamma^0 u^s(p) = tr[(\sum_s u^s(p)\overline{u}^s(p))\gamma^0(\sum_{s'}u^{s'}(p')\overline{u}^{s'}(p'))\gamma^0]$$

and using $\not{p} = \gamma\cdot p = \gamma^\mu p_\mu$:

$$\sum_s u^s(p)\overline{u}^s(p) = \not{p} + m$$

Thus substituting $\alpha = \frac{e^2}{4\pi}$:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{|\mathbf{p'}-\mathbf{p}|^4}\frac{1}{2}tr[(\not{p}+m)\gamma^0(\not{p}'+m)\gamma^0]$$

Since only products of even numbers of γ matrices have non-zero trace:

$$tr[(\not{p}+m)\gamma^0(\not{p}'+m)\gamma^0] = tr[\not{p}\gamma^0\not{p}'\gamma^0] + m^2 tr[\gamma^0\gamma^0]$$

from the identities:

$$tr(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$$

$$tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$$

we find:

$$tr[\not{p}\gamma^0\not{p}'\gamma^0] + m^2 tr[\gamma^0\gamma^0] = 4(2E^2 - p\cdot p' + m^2) = 4(2m^2 + |\mathbf{p}|^2(1+\cos\theta))$$

we also have:

$$|\mathbf{p'}-\mathbf{p}|^4 = (2|\mathbf{p}|^2(1-\cos\theta))^2 = 4|\mathbf{p}|^4(1-\cos\theta)^2$$

Now, $(1+\cos\theta) = 2\cos^2(\theta/2)$ and $(1-\cos\theta) = 2\sin^2(\theta/2)$. Thus,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\mathbf{p}|^4}\frac{1}{\sin^4(\theta/2)}(m^2 + |\mathbf{p}|^2\cos^2(\theta/2))$$

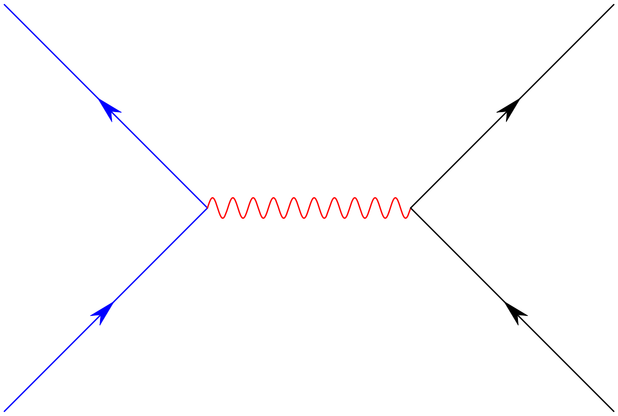
Using $|\mathbf{p}| = \gamma m\beta$ where $\gamma^2 = 1/(1-\beta^2)$ we find:

$$m^2 + |\mathbf{p}|^2\cos^2(\theta/2) = m^2\gamma^2(\gamma^{-2} + \beta^2(1-\sin^2(\theta/2))) = m^2\gamma^2(1-\beta^2\sin^2(\theta/2))$$

Thus, finally, we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\mathbf{p}|^2\beta^2}\frac{(1-\beta^2\sin^2(\theta/2))}{\sin^4(\theta/2)}$$

We now consider the process (where black lines are muons):



We denote the incoming and outgoing electron momenta by p and p' respectively. We denote the incoming and outgoing muon momenta by k and k' respectively. For convenience, the electron and muon spinors will be denoted $\mathbf{e}$ and μ respectively. Thus $(\not{p} + m_e)\mathbf{e}^s(p) = 0$ and $(\not{k} + m_\mu)\mu^s(k)$ where s labels the polarization of the spinors. We begin with the expression for the differential cross section for two particle to two particle scattering:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4E_pE_k\,|\upsilon_p - \upsilon_k|}\,\frac{|\mathbf{p}'|}{4(E_p+E_k)(2\pi)^2}$$

$$\times\frac{1}{4}\sum_{ss'}\sum_{rr'}\left|\mathcal{M}((\mathbf{e}^s(p),\mu^r(k)) \to (\mathbf{e}^{s'}(p'),\mu^{r'}(k')))\right|^2$$

Since this expression is invariant under boost along the beam axis we may work in the rest frame of the incoming muon. In this frame $\upsilon_k = 0$ and $E_k = m_\mu$. Furthermore, since we will eventually take $\lim_{m_\mu\to\infty}\frac{d\sigma}{d\Omega}$, we may freely impose this limit at any convenient point in the calculation. In particular since $p + k = p' + k'$, $((p-p')+k)^2 = m_\mu^2 = 2m_e^2 + m_\mu^2 - 2m_\mu(E_p - E_{p'})$ and therefore $E_p = E_{p'}$ and $|\mathbf{p}| = |\mathbf{p}'|$ as $m_\mu \to \infty$. Thus we have:

$$\frac{d\sigma}{d\Omega} = \frac{1}{16m_\mu(E_p+m_\mu)}\,\frac{|\mathbf{p}|}{E_p\,|\upsilon_p|}\,\frac{1}{(2\pi)^2}$$

$$\times\frac{1}{4}\sum_{ss'}\sum_{rr'}\left|\mathcal{M}(\mathbf{e}^s(p)\,\mu^r(k)\ \to\ \mathbf{e}^{s'}(p')\,\mu^{r'}(k'))\right|^2$$

Since $|\mathbf{p}| = |\upsilon_p|\,E_p$ and for $m_\mu \gg m_e$ $m_\mu(E_p+m_\mu) = m_\mu^2(1+E_p/m_\mu) \approx m_\mu^2$:

$$\frac{d\sigma}{d\Omega} = \frac{1}{16(2\pi)^2m_\mu^2}\frac{1}{4}\sum_{ss'}\sum_{rr'}\left|\mathcal{M}(\mathbf{e}^s(p)\,\mu^r(k)\ \to\ \mathbf{e}^{s'}(p')\,\mu^{r'}(k'))\right|^2$$

Where, invoking the Feynman rules:

$$i\mathcal{M} = i\mathcal{M}(\mathbf{e}^s(p)\,\mu^r(k)\ \to\ \mathbf{e}^{s'}(p')\,\mu^{r'}(k')) = \frac{ie^2}{(p-p')^2}\,\overline{\mathbf{e}}^{s'}(p')\gamma^\alpha\mathbf{e}^s(p)\,\overline{\mu}^{r'}(k')\gamma_\alpha\mu^r(k)$$

Thus using the spin sum rules given above:

$$\sum_{spins}|\mathcal{M}|^2 = (\frac{e^2}{(p-p')^2})^2\,tr[(\not{p}+m_e)\gamma^\alpha(\not{p}'+m_e)\gamma^\beta]\,tr[(\not{k}+m_\mu)\gamma_\alpha(\not{k}'+m_\mu)\gamma_\beta]$$

using

$$tr[(\not{p}+m_e)\gamma^\alpha(\not{p}'+m_e)\gamma^\beta] = tr[(\not{p}\gamma^\alpha\not{p}'\gamma^\beta] + m_e^2tr[\gamma^\alpha\gamma^\beta]$$

and the above identities:

$$tr[(\not{p}+m_e)\gamma^\alpha(\not{p}'+m_e)\gamma^\beta]=4(p^\alpha p'^\beta+p'^\alpha p^\beta-\eta^{\alpha\beta}(p\cdot p'-m_e^2))$$

thus, after some rearrangements:

$$\sum_{spins}|\mathcal{M}|^2=(\frac{e^2}{(p-p')^2})^2\,32[k\cdot p\,k'\cdot p'+p'\cdot k\,p\cdot k'$$

$$-k\cdot k'(p\cdot p'-m_e^2)-p\cdot p'(k\cdot k'-m_\mu^2)$$

$$+2(p\cdot p'-m_e^2)(k\cdot k'-m_\mu^2)]$$

using $k\cdot k'=m_\mu E_{k'}\approx m_\mu^2$ and factoring m_μ^2 out of $\sum|\mathcal{M}|^2$:

$$\frac{d\sigma}{d\Omega}=\frac{2}{(2\pi)^2}\frac{1}{4}(\frac{e^2}{(p-p')^2})^2[m_e^2-p\cdot p'+(k\cdot p\,k'\cdot p'+p'\cdot k\,p\cdot k')/m_\mu^2]$$

Now $k'\cdot p'/m_\mu=(p+k-p')\cdot p'/m_\mu\approx k\cdot p'/m_\mu=E_{p'}$. Similarly $k'\cdot p/m_\mu\approx E_p\approx E_{p'}$. Thus we have:

$$\frac{d\sigma}{d\Omega}=\frac{2}{(2\pi)^2}\frac{1}{4}(\frac{e^2}{(p-p')^2})^2\{m_e^2-p\cdot p'+2E_p^2]$$

Again, as above, using $|\mathbf{p}|=|\mathbf{p}'|=\gamma m_e\beta$ where $\gamma^2=1/(1-\beta^2)$ we find:

$$(m_e^2-p\cdot p'+2E_p^2)=(2m_e^2+|\mathbf{p}|^2(1+\cos\theta))=2m_e^2\gamma^2(1-\beta^2\sin^2(\theta/2))$$

Also substituting $\alpha=\frac{e^2}{4\pi}$ and using $(p-p')^2=|\mathbf{p}-\mathbf{p}'|^2$since $E_p\approx E_{p'}$:

$$\frac{d\sigma}{d\Omega}=\frac{4\alpha^2}{|\mathbf{p}-\mathbf{p}'|^4}m_e^2\gamma^2(1-\beta^2\sin^2(\theta/2))$$

Using the above result

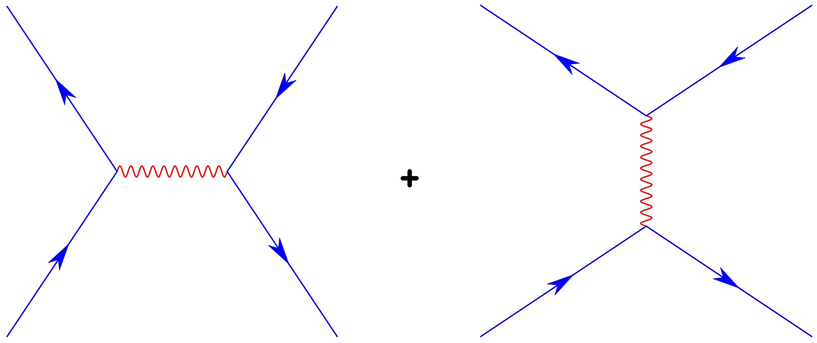
$$|\mathbf{p}'-\mathbf{p}|^4=4\,|\mathbf{p}|^4\,(1-\cos\theta)^2=16\,|\mathbf{p}|^4\sin^4(\theta/2)$$

We see again, finally:

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{4\,|\mathbf{p}|^2\,\beta^2}\frac{(1-\beta^2\sin^2(\theta/2))}{\sin^4(\theta/2)}$$

2.) P&S 5.2: Bhabha Scattering

We consider the process given by the sum of the two diagrams:



This is the amplitude for electron-positron scattering to O(α^2).

To calculate the amplitude associated with the diagram we turn to the operator formalism. We will verify the result given using Feynman rules. We denote incoming electron and positron momenta by p_1 and k_1 respectively. We similarly denote outgoing electron and positron momenta by p_2 and k_2 respectively. The amplitude will be expressed in terms of the incoming and outgoing states:

$$\left|\mathbf{e}^-_{p_1 s_1}\ \mathbf{e}^+_{k_1 r_1}\right\rangle = \sqrt{2E_{p_1}}\sqrt{2E_{k_1}}\, a^\dagger_{p_1 s_1} b^\dagger_{k_1 r_1} \left|0\right\rangle$$

$$\left|\mathbf{e}^-_{p_2 s_2}\ \mathbf{e}^+_{k_2 r_2}\right\rangle = \sqrt{2E_{p_2}}\sqrt{2E_{k_2}}\, a^\dagger_{p_2 s_2} b^\dagger_{k_2 r_2} \left|0\right\rangle$$

We evaluate:

$$\mathcal{A} = i\mathcal{M}(\left|\mathbf{e}^-_{p_1 s_1}\ \mathbf{e}^+_{k_1 r_1}\right\rangle \to \left|\mathbf{e}^-_{p_2 s_2}\ \mathbf{e}^+_{k_2 r_2}\right\rangle)\ (2\pi)^4\delta((p_2+k_2)-(p_1+k_1))$$

$$= \left\langle \mathbf{e}^-_{p_2 s_2}\ \mathbf{e}^+_{k_2 r_2}\right| (T\{\exp(-i\int d^4x \mathcal{H}_I(x))\} - 1) \left|\mathbf{e}^-_{p_1 s_1}\ \mathbf{e}^+_{k_1 r_1}\right\rangle$$

Where, for QED

$$\mathcal{H}_I(x) = e\overline{\psi}\not{A}\psi$$

To O(α^2):

$$\mathcal{A} = \frac{(-ie)^2}{2!}\int d^4x d^4y \left\langle \mathbf{e}^-_{p_2 s_2}\ \mathbf{e}^+_{k_2 r_2}\right| T\{\overline{\psi}(x)\not{A}(x)\psi(x)\overline{\psi}(y)\not{A}(y)\psi(y)\} \left|\mathbf{e}^-_{p_1 s_1}\ \mathbf{e}^+_{k_1 r_1}\right\rangle$$

$$\mathcal{A} = \frac{(-ie)^2}{2!}\int d^4x d^4y \left\langle 0\right| T\{A_\mu(x)A_\nu(y)\} \left|0\right\rangle$$
$$\times \left\langle \mathbf{e}^-_{p_2 s_2}\ \mathbf{e}^+_{k_2 r_2}\right| T\{\overline{\psi}(x)\gamma^\mu\psi(x)\overline{\psi}(y)\gamma^\nu\psi(y)\} \left|\mathbf{e}^-_{p_1 s_1}\ \mathbf{e}^+_{k_1 r_1}\right\rangle$$

Ignoring vacuum bubbles and using Wicks Theorem:

$$\mathcal{A} = (-ie)^2\int d^4x d^4y \left\langle 0\right| T\{A_\mu(x)A_\nu(y)\} \left|0\right\rangle \gamma^\mu_{ab}\gamma^\nu_{cd}\sqrt{2E_{p_1}2E_{k_1}2E_{p_2}2E_{k_2}}$$

$$\times \Big[\left\langle 0\right| a_{p_2 s_2}\overline{\psi}_a(x)\left|0\right\rangle \left\langle 0\right| b_{k_2 r_2}\psi_b(x)\left|0\right\rangle \left\langle 0\right| \overline{\psi}_c(y) b^\dagger_{k_1 r_1}\left|0\right\rangle \left\langle 0\right| \psi_d(y) a^\dagger_{p_1 s_1}\left|0\right\rangle$$

$$- \left\langle 0\right| a_{p_2 s_2}\overline{\psi}_c(y)\left|0\right\rangle \left\langle 0\right| b_{k_2 r_2}\psi_b(x)\left|0\right\rangle \left\langle 0\right| \overline{\psi}_a(x) b^\dagger_{k_1 r_1}\left|0\right\rangle \left\langle 0\right| \psi_d(y) a^\dagger_{p_1 s_1}\left|0\right\rangle\Big]$$

Where,

$$\left\langle 0\right| a_{p_2 s_2}\overline{\psi}_a(x)\left|0\right\rangle = \frac{\exp(ip_2\cdot x)}{\sqrt{2E_{p_2}}}\overline{u}^{s_2}_a(p_2)$$

$$\langle 0|\, b_{k_2 r_2}\psi_b(x)\,|0\rangle = \frac{\exp(ik_2\cdot x)}{\sqrt{2E_{k_2}}}\upsilon_b^{r_2}(k_2)$$

$$\langle 0|\, \overline{\psi}_c(y) b^\dagger_{k_1 r_1}\,|0\rangle = \frac{\exp(-ik_1\cdot y)}{\sqrt{2E_{k_1}}}\overline{\upsilon}_c^{r_1}(k_1)$$

$$\langle 0|\, \psi_d(y) a^\dagger_{p_1 s_1}\,|0\rangle = \frac{\exp(-ip_1\cdot y)}{\sqrt{2E_{p_1}}}u_d^{s_1}(p_1)$$

Thus,

$$\mathcal{A} = (-ie)^2\int d^4x d^4y\,\langle 0|\, T\{A_\mu(x)A_\nu(y)\}\,|0\rangle$$
$$\times\,[\exp(-i(p_1+k_1)\cdot y + i(p_2+k_2)\cdot x))\overline{u}^{s_2}(p_2)\gamma^\mu\upsilon^{r_2}(k_2)\overline{\upsilon}^{r_1}(k_1)\gamma^\nu u^{s_1}(p_1)$$
$$-\exp(-i(p_1-p_2)\cdot y + i(k_2-k_1)\cdot x))\overline{\upsilon}^{r_1}(k_1)\gamma^\mu\upsilon^{r_2}(k_2)\overline{u}^{s_2}(p_2)\gamma^\nu u^{s_1}(p_1)]$$

Now,

$$\langle 0|\, T\{A_\mu(x)A_\nu(y)\}\,|0\rangle = \int\frac{d^4q}{(2\pi)^4}\frac{-i\eta_{\mu\nu}}{q^2+i\epsilon}\exp(-iq\cdot(x-y))$$

Thus,

$$\mathcal{A} = (-ie)^2(2\pi)^4\int d^4q\frac{-i\eta_{\mu\nu}}{q^2+i\epsilon}$$
$$\times\,[\delta(q-(p_1+k_1))\delta(q-(p_2+k_2))\overline{u}^{s_2}(p_2)\gamma^\mu\upsilon^{r_2}(k_2)\overline{\upsilon}^{r_1}(k_1)\gamma^\nu u^{s_1}(p_1)$$
$$-\,\delta(q-(p_1-p_2))\delta(q-(k_2-k_1)))\overline{\upsilon}^{r_1}(k_1)\gamma^\mu\upsilon^{r_2}(k_2)\overline{u}^{s_2}(p_2)\gamma^\nu u^{s_1}(p_1)]$$

$$\mathcal{A} = (-ie)^2(2\pi)^4\delta((p_2+k_2)-(p_1+k_1))$$
$$\times\left[\frac{-i\eta_{\mu\nu}}{(p_1+k_1)^2+i\epsilon}\overline{u}^{s_2}(p_2)\gamma^\mu\upsilon^{r_2}(k_2)\overline{\upsilon}^{r_1}(k_1)\gamma^\nu u^{s_1}(p_1)\right.$$
$$\left.-\,\frac{-i\eta_{\mu\nu}}{(p_1-p_2)^2+i\epsilon}\overline{\upsilon}^{r_1}(k_1)\gamma^\mu\upsilon^{r_2}(k_2)\overline{u}^{s_2}(p_2)\gamma^\nu u^{s_1}(p_1)\right]$$

This is exactly the result we would get from the Feynman rules where we would write down:

$$i\mathcal{M} = ie^2\eta_{\mu\nu}\left[\frac{\overline{u}^{s_2}(p_2)\gamma^\mu\upsilon^{r_2}(k_2)\overline{\upsilon}^{r_1}(k_1)\gamma^\nu u^{s_1}(p_1)}{(p_1+k_1)^2+i\epsilon}\right.$$
$$\left.-\frac{\overline{\upsilon}^{r_1}(k_1)\gamma^\mu\upsilon^{r_2}(k_2)\overline{u}^{s_2}(p_2)\gamma^\nu u^{s_1}(p_1)}{(p_1-p_2)^2+i\epsilon}\right]$$

Note that the minus sign results because the diagrams are equivalent up to an exchange of an outgoing electron with an incoming positron or vice-versa. The minus sign is due to the Fermi exchange statistics. We would like to compute the differential cross section in the limit $m_e\to 0$. Since all particles have identical masses we have:

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi(p_1+k_1)^2}\frac{1}{4}\sum_{s_1 s_2}\sum_{r_1 r_2}\left|\mathcal{M}(\left|\mathbf{e}^-_{p_1 s_1}\ \mathbf{e}^+_{k_1 r_1}\right\rangle\to\left|\mathbf{e}^-_{p_2 s_2}\ \mathbf{e}^+_{k_2 r_2}\right\rangle)\right|^2$$

Ignoring the $i\epsilon$ prescription and denoting $\upsilon_j = \upsilon^{r_j}(k_j)$ and $u_j = u^{s_j}(p_j)$:

$$\sum_{spins}|\mathcal{M}|^2 = e^4\eta_{\mu\nu}\eta_{\rho\sigma}\sum_{spins}\left[\left(\frac{\overline{u}_2\gamma^\mu\upsilon_2\overline{\upsilon}_1\gamma^\nu u_1}{(p_1+k_1)^2}-\frac{\overline{\upsilon}_1\gamma^\mu\upsilon_2\overline{u}_2\gamma^\nu u_1}{(p_1-p_2)^2}\right)\right.$$
$$\left.\times\left(\frac{\overline{\upsilon}_2\gamma^\rho u_2\overline{u}_1\gamma^\sigma\upsilon_1}{(p_1+k_1)^2}-\frac{\overline{\upsilon}_2\gamma^\rho\upsilon_1\overline{u}_1\gamma^\sigma u_2}{(p_1-p_2)^2}\right)\right]$$

Using the spin sum relations given above and taking $m_e \to 0$:

$$\begin{aligned}\sum_{spins} |\mathcal{M}|^2 = e^4 \eta_{\mu\nu}\eta_{\rho\sigma} \Bigg[& \frac{1}{(p_1+k_1)^4} tr[\not{p}_2\gamma^\mu \not{k}_2\gamma^\rho]\; tr[\not{k}_1\gamma^\nu \not{p}_1\gamma^\sigma] \\ & - \frac{1}{(p_1+k_1)^2(p_1-p_2)^2} tr[\not{p}_2\gamma^\mu \not{k}_2\gamma^\rho \not{k}_1\gamma^\nu \not{p}_1\gamma^\sigma] \\ & - \frac{1}{(p_1+k_1)^2(p_1-p_2)^2} tr[\not{k}_1\gamma^\mu \not{k}_2\gamma^\rho \not{p}_2\gamma^\nu \not{p}_1\gamma^\sigma] \\ & + \frac{1}{(p_1-p_2)^4} tr[\not{k}_1\gamma^\mu \not{k}_2\gamma^\rho]\; tr[\not{p}_2\gamma^\nu \not{p}_1\gamma^\sigma] \Bigg] \end{aligned}$$

$$= e^4 \left[\frac{\Gamma_A(p_1,k_1,p_2,k_2)}{(p_1+k_1)^4} - \frac{\Gamma_B(p_1,k_1,p_2,k_2)+\Gamma_B(p_1,p_2,k_1,k_2)}{(p_1+k_1)^2(p_1-p_2)^2} + \frac{\Gamma_A(p_1,p_2,k_1,k_2)}{(p_1-p_2)^4} \right]$$

Where:

$$\Gamma_A(p_1,k_1,p_2,k_2) = tr[\not{p}_2\gamma^\mu \not{k}_2\gamma^\nu]\; tr[\not{k}_1\gamma_\mu \not{p}_1\gamma_\nu]$$

$$\Gamma_B(p_1,k_1,p_2,k_2) = tr[\not{p}_2\gamma^\mu \not{k}_2\gamma^\nu \not{k}_1\gamma_\mu \not{p}_1\gamma_\nu]$$

Since, from above, $tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$, we have:

$$tr[\not{p}_2\gamma^\mu \not{k}_2\gamma^\nu] = 4(p_2^\mu k_2^\nu + k_2^\mu p_2^\nu - p_2\cdot k_2\eta^{\mu\nu})$$

Thus,

$$\Gamma_A(p_1,k_1,p_2,k_2) = 32(p_2\cdot k_1 k_2\cdot p_1 + p_1\cdot p_2 k_1\cdot k_2)$$

Note that since $p_j^2 = k_j^2 = 0$ and $p_1 + k_1 = p_2 + k_2$:

$$s = (p_1+k_1)^2 = 2p_1\cdot k_1 = 2p_2\cdot k_2$$

$$t = (p_1-p_2)^2 = -2p_1\cdot p_2 = -2k_1\cdot k_2$$

$$u = (k_2-p_1)^2 = -2p_1\cdot k_2 = -2k_1\cdot p_2$$

Thus, $\Gamma_A(p_1,k_1,p_2,k_2) = 8(u^2+t^2)$ and $\Gamma_A(p_1,p_2,k_1,k_2) = 8(u^2+s^2)$.

Using $\{\gamma^\mu, \not{p}\} = 2p^\mu$ we have:

$$\Gamma_B(p_1,k_1,p_2,k_2) = tr[\not{p}_2\gamma^\mu \not{k}_2\gamma^\nu \not{k}_1\gamma_\mu \not{p}_1\gamma_\nu]$$

$$\begin{aligned} &= 2tr[\not{p}_2\gamma^\mu \not{k}_2\gamma_\mu \not{p}_1 \not{k}_1] - tr[\not{p}_2\gamma^\mu \not{k}_2 \not{k}_1\gamma^\nu\gamma_\mu \not{p}_1\gamma_\nu] \\ &= 4tr[\not{p}_2 \not{k}_2 \not{p}_1 \not{k}_1] - 2tr[\not{p}_2\gamma^\mu\gamma_\mu \not{k}_2 \not{p}_1 \not{k}_1] \\ &\quad -2tr[\not{p}_2 \not{p}_1 \not{k}_2 \not{k}_1\gamma^\nu\gamma_\nu] + tr[\not{p}_2\gamma^\mu \not{k}_2 \not{k}_1\gamma^\nu \not{p}_1\gamma_\mu\gamma_\nu] \\ &= -4tr[\not{p}_2 \not{k}_2 \not{p}_1 \not{k}_1] - 8tr[\not{p}_2 \not{p}_1 \not{k}_2 \not{k}_1] \\ &\quad -tr[\not{p}_2\gamma^\mu \not{k}_2 \not{k}_1\gamma^\nu \not{p}_1\gamma_\nu\gamma_\mu] + 2tr[\not{p}_2\gamma^\mu \not{k}_2 \not{k}_1\gamma_\mu \not{p}_1] \\ &= -4tr[\not{p}_2 \not{k}_2 \not{p}_1 \not{k}_1] - 4tr[\not{p}_2 \not{p}_1 \not{k}_2 \not{k}_1] \end{aligned}$$

From the above identity:

$$tr[\not{p}_2 \not{k}_2 \not{p}_1 \not{k}_1] = 4(p_1\cdot k_1 p_2\cdot k_2 - p_1\cdot p_2 k_1\cdot k_2 + p_1\cdot k_2 p_2\cdot k_1)$$

$$tr[\not{p}_2 \not{p}_1 \not{k}_2 \not{k}_1] = 4(p_1\cdot p_2 k_1\cdot k_2 - p_1\cdot k_1 p_2\cdot k_2 + p_1\cdot k_2 p_2\cdot k_1)$$

$$\Gamma_B(p_1,k_1,p_2,k_2) = -32(p_1\cdot k_2 p_2\cdot k_1) = -8u^2$$

This leads to:

$$\sum_{spins} |\mathcal{M}|^2 = e^4 \left[\frac{8(u^2+t^2)}{s^2} + \frac{16u^2}{st} + \frac{8(u^2+s^2)}{t^2} \right]$$

Finally:

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{16\pi s} e^4 \left[\frac{(u^2+t^2)}{s^2} + \frac{2u^2}{st} + \frac{(u^2+s^2)}{t^2} \right]$$

Rearranging and substituting $\alpha = \frac{e^2}{4\pi}$:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s} \left[u^2(\frac{1}{s} + \frac{1}{t})^2 + (\frac{t}{s})^2 + (\frac{s}{t})^2 \right]$$

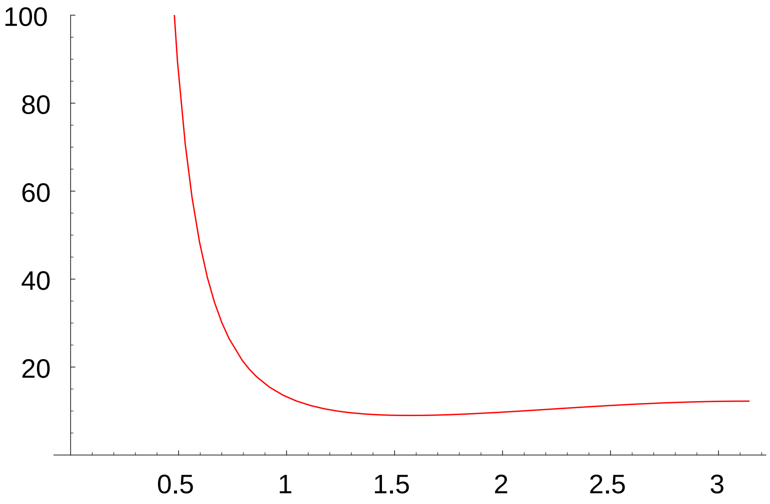
Since the particles are massless $u + t + s = 0$ and therefore we may write:

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s} \left[3 + 2(\frac{t}{s} + \frac{s}{t}) + (\frac{t}{s})^2 + (\frac{s}{t})^2 \right]$$

Working in the center of momentum frame where $E_p = E_{p_j} = E_{k_j}$ and setting $s = 2p_1 \cdot k_1 = 2E_p^2$ and $t = -2p_1 \cdot p_2 = -2E_p^2(1-\cos\theta)$ Thus,

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{E_p^2} \left[3 + 2((1-\cos\theta) + (1-\cos\theta)^{-1}) + (1-\cos\theta)^2 + (1-\cos\theta)^{-2} \right]$$

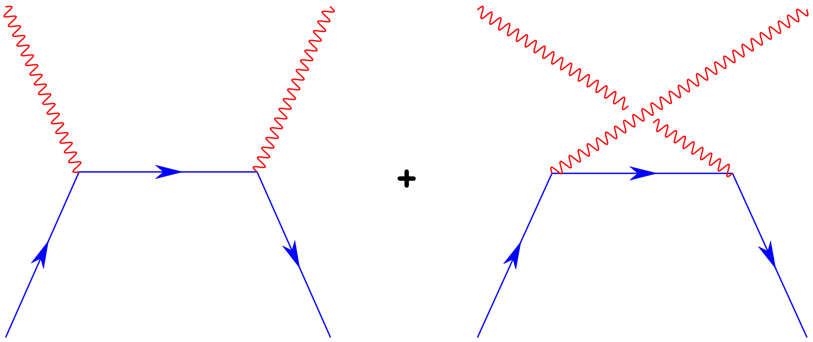
The following is the graph of the function $f(\theta) = (E_p^2/(\pi\alpha^2))\frac{d\sigma}{d\cos\theta}$:



The differential cross section diverges as $\theta \rightarrow 0$ because in the first (t channel) graph the photon propagator diverges as the photon approaches its physical mass.

3.) P&S 5.4(a): Positronium Lifetimes

To compute the rate of decay of positronium into two photons consider the process:



Using the Feynman rules and denoting incoming electron and positron momenta and spins by (p, s) and (p', s') respectively and denoting outgoing photon momenta and polarizations by (k, α) and (k', α') respectively we have:

$$i\mathcal{M} = (-ie)^2(\varepsilon^{\alpha}_{\mu}(\widehat{\mathbf{k}}))^*(\varepsilon^{\alpha'}_{\nu}(\widehat{\mathbf{k}}'))^*$$

$$\times\left[\overline{v}^{s'}(p')\gamma^{\mu}\frac{i(\not p - \not k + m)}{(p-k)^2 - m^2}\gamma^{\nu}u^s(p) + \overline{v}^{s'}(p')\gamma^{\nu}\frac{i(\not k - \not p' + m)}{(k-p')^2 - m^2}\gamma^{\mu}u^s(p)\right]$$

Now,

$$(\not p + m)\gamma^{\nu}u^s(p) = -\gamma^{\nu}(\not p - m)u^s(p) + 2p^{\nu}u^s(p) = 2p^{\nu}u^s(p)$$

$$\overline{v}^{s'}(p')\gamma^{\nu}(-\not p' + m) = \overline{v}^{s'}(p')(\not p' + m)\gamma^{\nu} - 2\overline{v}^{s'}(p')p'^{\nu} = -2\overline{v}^{s'}(p')p'^{\nu}$$

Where we have used the Dirac equation. We represent the polarization vectors in terms of a pure rotation $\mathcal{R}_{\mu}^{\ \nu}(\widehat{\mathbf{z}} \to \widehat{\mathbf{k}})$ that brings the unit vector $\widehat{\mathbf{z}}$ into the unit vector $\widehat{\mathbf{k}}$. Since $\widehat{\mathbf{k}}' = -\widehat{\mathbf{k}}$ in the center of momentum frame, we have:

$$\varepsilon^{\alpha}_{\mu}(\widehat{\mathbf{k}}) = \mathcal{R}_{\mu}^{\ \nu}(\widehat{\mathbf{z}} \to \widehat{\mathbf{k}})\varepsilon^{\alpha}_{\nu}(\widehat{\mathbf{z}}) \equiv \mathcal{R}_{\mu}^{\ \nu}\varepsilon^{\alpha}_{\nu}$$

$$\varepsilon^{\alpha'}_{\mu}(\widehat{\mathbf{k}}') = \mathcal{R}_{\mu}^{\ \nu}(\widehat{\mathbf{z}} \to -\widehat{\mathbf{k}})\varepsilon^{\alpha'}_{\nu}(\widehat{\mathbf{z}}) = -\mathcal{R}_{\mu}^{\ \nu}\varepsilon^{\alpha'}_{\nu}$$

We are working in the non-relativistic limit so that:

$$p = (E_{\mathbf{p}}, |\mathbf{p}|\,\widehat{\mathbf{z}}) \approx (m, 0) \ \ p' = (E_{\mathbf{p}}, -\,|\mathbf{p}|\,\widehat{\mathbf{z}}) \approx (m, 0)$$

$$k = (E_{\mathbf{p}}, E_{\mathbf{p}}\widehat{\mathbf{k}}) \approx (m, m\widehat{\mathbf{k}}) \ \ k = (E_{\mathbf{p}}, -E_{\mathbf{p}}\widehat{\mathbf{k}}) \approx (m, -m\widehat{\mathbf{k}})$$

$$(p-k)^2 - m^2 \approx (k-p')^2 - m^2 \approx -2m^2$$

We also choose to work in a circularly polarized basis for the photon states :

$$\varepsilon^{+} = (0, (\widehat{\mathbf{x}} + i\widehat{\mathbf{y}})/\sqrt{2}))$$

$$\varepsilon^{-} = (0, (\widehat{\mathbf{x}} - i\widehat{\mathbf{y}})/\sqrt{2}))$$

Thus $p^{\nu}\varepsilon^{\alpha'}_{\nu}(\widehat{\mathbf{k}}') \approx (m, 0)^{\nu}\mathcal{R}_{\nu}^{\ \mu}\varepsilon^{\alpha'}_{\mu} = 0$ since $\varepsilon^{\alpha'}$ is purely spatial. Thus we have:

$$i\mathcal{M} = \frac{ie^2}{2m^2}(\mathcal{R}_{\mu}^{\ \rho}\varepsilon^{\alpha}_{\rho})^*(\mathcal{R}_{\nu}^{\ \sigma}\varepsilon^{\alpha'}_{\sigma})^*\,\overline{v}^{s'}(p')\left[\gamma^{\mu}\not k\gamma^{\nu} - \gamma^{\nu}\not k\gamma^{\mu}\right]u^s(p)$$

Now,

$$\mathcal{R}_{\mu}^{\ \rho}\gamma^{\mu} = (\mathcal{R}^{-1})^{\rho}_{\ \mu}\gamma^{\mu} = \Lambda(\mathcal{R})\gamma^{\rho}\Lambda(\mathcal{R})^{-1}$$

and, denoting $\Lambda \equiv \Lambda(\mathcal{R})$

$$\not{k} = k_\nu \gamma^\nu = m\mathcal{R}_\nu^\sigma (1, \widehat{\mathbf{z}})_\sigma \gamma^\nu = m(1, \widehat{\mathbf{z}})_\sigma \Lambda \gamma^\sigma \Lambda^{-1} = m\Lambda(\gamma^0 - \gamma^3)\Lambda^{-1}$$

Thus,

$$i\mathcal{M} = \frac{ie^2}{2m}(\varepsilon_\rho^\alpha)^*(\varepsilon_\sigma^{\alpha'})^*\, \overline{\upsilon}^{s'}(p')\Lambda\left[\gamma^\rho(\gamma^0-\gamma^3)\gamma^\sigma - \gamma^\sigma(\gamma^0-\gamma^3)\gamma^\rho\right]\Lambda^{-1}u^s(p)$$

We consider the operator:

$$\chi^{\alpha\alpha'} = (\varepsilon_\rho^\alpha)^*(\varepsilon_\sigma^{\alpha'})^*\left[\gamma^\rho(\gamma^0-\gamma^3)\gamma^\sigma - \gamma^\sigma(\gamma^0-\gamma^3)\gamma^\rho\right]$$

Since $(\varepsilon^\pm)^* = \varepsilon^\mp$ we have:

$$\chi^{++} = \left[\not{\varepsilon}^-(\gamma^0-\gamma^3)\not{\varepsilon}^- - \not{\varepsilon}^-(\gamma^0-\gamma^3)\not{\varepsilon}^-\right] = 0$$

$$\chi^{--} = \left[\not{\varepsilon}^+(\gamma^0-\gamma^3)\not{\varepsilon}^+ - \not{\varepsilon}^+(\gamma^0-\gamma^3)\not{\varepsilon}^+\right] = 0$$

$$\chi^{+-} = -\chi^{-+} = \left[\not{\varepsilon}^-(\gamma^0-\gamma^3)\not{\varepsilon}^+ - \not{\varepsilon}^+(\gamma^0-\gamma^3)\not{\varepsilon}^-\right]$$

We write:

$$i\mathcal{M} = \frac{ie^2}{2m}\overline{\upsilon}^{s'}(p')\Lambda\chi^{\alpha\alpha'}\Lambda^{-1}u^s(p)$$

Since $\not{\varepsilon}^\pm = (\gamma^1 \pm i\gamma^2)/\sqrt{2}$:

$$\chi^{+-} = \frac{1}{2}\left[(\gamma^1 - i\gamma^2)(\gamma^0-\gamma^3)(\gamma^1+i\gamma^2) - (\gamma^1+i\gamma^2)(\gamma^0-\gamma^3)(\gamma^1-i\gamma^2)\right]$$

$$\chi^{+-} = \frac{-1}{2}(\gamma^0-\gamma^3)\left[(\gamma^1-i\gamma^2)(\gamma^1+i\gamma^2) - (\gamma^1+i\gamma^2)(\gamma^1-i\gamma^2)\right]$$

$$\chi^{+-} = -i(\gamma^0-\gamma^3)(\gamma^1\gamma^2 - \gamma^2\gamma^1) = -i(\gamma^0-\gamma^3)\left[\gamma^1,\gamma^2\right]$$

Denoting $\sigma = (1, \overrightarrow{\sigma})$ and $\overline{\sigma} = (1, -\overrightarrow{\sigma})$ we have:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}$$

thus, using $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$

$$\chi^{+-} = i\begin{pmatrix} 0 & (1-\sigma^3) \\ (1+\sigma^3) & 0 \end{pmatrix}\begin{pmatrix} [\sigma^1,\sigma^2] & 0 \\ 0 & [\sigma^1,\sigma^2] \end{pmatrix}$$

$$\chi^{+-} = -2\begin{pmatrix} 0 & (\sigma^3-1) \\ (\sigma^3+1) & 0 \end{pmatrix}$$

Since $u^s(p) = \Lambda u^s(\mathcal{R}^{-1}p)$ and $\overline{\upsilon}^{s'}(p') = \overline{\upsilon}^{s'}(\mathcal{R}^{-1}p')\Lambda^{-1}$ we have:

$$i\mathcal{M} = \frac{ie^2}{2m}\overline{\upsilon}^{s'}(\mathcal{R}^{-1}p')\chi^{\alpha\alpha'}u^s(\mathcal{R}^{-1}p)$$

Now, since we are working in the non-relativistic limit:

$$u^s(\mathcal{R}^{-1}p) = u^s(p) = \sqrt{m}\begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}$$

$$\upsilon^{s'}(\mathcal{R}^{-1}p') = \upsilon^{s'}(p') = \sqrt{m}\begin{pmatrix} \zeta^{s'} \\ -\zeta^{s'} \end{pmatrix}$$

Thus, placing indices on the matrix element:

$$i\mathcal{M}^{\alpha\alpha'} = \frac{ie^2}{2m}\overline{v}^{s'}(p')\chi^{\alpha\alpha'}u^s(p)$$

$$i\mathcal{M}^{+-} = -i\mathcal{M}^{-+} = -ie^2 \begin{pmatrix} -\zeta^{s'\dagger} & \zeta^{s'\dagger} \end{pmatrix} \begin{pmatrix} 0 & (\sigma^3 - 1) \\ (\sigma^3 + 1) & 0 \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}$$

$$i\mathcal{M}^{+-} = -i\mathcal{M}^{-+} = -2ie^2\zeta^{s'\dagger}\xi^s = -2ie^2 tr(\xi^s\zeta^{s'\dagger})$$

Where $i\mathcal{M}^{++} = i\mathcal{M}^{--} = 0$. We now consider the matrix $\lambda^{ss'} = \xi^s\zeta^{s'\dagger}$ where we use basis vectors for spin along the $\widehat{\mathbf{z}}$ axis.

Thus, using,

$$\xi^+ = -\zeta^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi^- = \zeta^+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we have:

$$\lambda^{++} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \lambda^{--} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$\lambda^{+-} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \lambda^{-+} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Choosing a basis for total spin eigenstates $\widetilde{\lambda}^{js}$:

$$\widetilde{\lambda}^{1+} = \lambda^{++} \quad \widetilde{\lambda}^{1-} = \lambda^{--}$$

$$\widetilde{\lambda}^{10} = (\lambda^{+-} + \lambda^{-+})/\sqrt{2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\widetilde{\lambda}^{00} = (\lambda^{+-} - \lambda^{-+})/\sqrt{2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

We see that all of the $\widetilde{\lambda}^{1s}$ matrices are traceless and thus will not contribute to the amplitude. Thus only the spin 0 state of positronium will lead to two-photon decay at order $\mathrm{O}(\alpha^2)$. We rewrite the amplitude for the transition with the understanding that only the spin 0 state makes a contribution:

$$i\mathcal{M}^{+-} = -i\mathcal{M}^{-+} = -2ie^2 tr(\widetilde{\lambda}^{00}) = 2\sqrt{2}\, ie^2$$

We now consider the spin 0 positronium state vector:

$$|B\rangle = \sqrt{2(2m)}\int \frac{d^3p}{(2\pi)^3}\widetilde{\psi}(\mathbf{p})\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}(|\mathbf{p}\uparrow, -\mathbf{p}\downarrow\rangle - |\mathbf{p}\downarrow, -\mathbf{p}\uparrow\rangle)/\sqrt{2}$$

Where $\widetilde{\psi}(\mathbf{p})$ is the Fourier transform of the non-relativistic ground state hydrogen wave function:

$$\widetilde{\psi}(\mathbf{p}) = \int d^3x\, e^{i\mathbf{p}\cdot\mathbf{x}}\,\psi(\mathbf{x}) \quad \psi(\mathbf{x}) = \frac{(m\alpha)^{3/2}}{\sqrt{\pi}}\exp(-\alpha m\,|\mathbf{x}|)$$

Thus,

$$i\widetilde{\mathcal{M}}^{\alpha\alpha'}(B \to 2\gamma) = i\mathcal{M}^{\alpha\alpha'}\frac{1}{\sqrt{2m}}\int \frac{d^3p}{(2\pi)^3}\widetilde{\psi}(\mathbf{p}) = i\mathcal{M}^{\alpha\alpha'}\frac{\psi(0)}{\sqrt{2m}}$$

Summing over final photon polarizations and averaging over initial positronium states (only one of which contributes to the amplitude for decay) the decay rate is:

$$\Gamma(B \to 2\gamma) = \frac{1}{2(2m)}\frac{1}{4}\sum_{\alpha\alpha'}\int d\Omega\frac{1}{16\pi^2}\frac{|\mathbf{k}|}{2m}\left|\widetilde{\mathcal{M}}^{\alpha\alpha'}\right|^2$$

or, since the amplitude has no angular dependence:

$$\Gamma(B \to 2\gamma) = \frac{1}{2(2m)} \frac{1}{4} \frac{|\psi(0)|^2}{16\pi m} \sum_{\alpha\alpha'} \left|\mathcal{M}^{\alpha\alpha'}\right|^2$$

Now, from above:

$$\sum_{\alpha\alpha'} \left|\mathcal{M}^{\alpha\alpha'}\right|^2 = \left|\mathcal{M}^{+-}\right|^2 + \left|\mathcal{M}^{-+}\right|^2 = 8e^4 = 8(4\pi)^2\alpha^2$$

$$|\psi(0)|^2 = (m\alpha)^3/\pi$$

Thus, finally,

$$\Gamma = \frac{1}{2} m\alpha^5$$