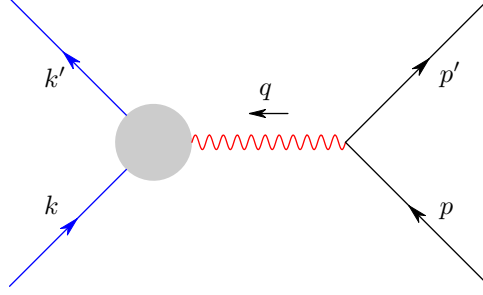


QFT 2 : Problem Set 1

1.) Peskin & Schroeder 6.1

We consider the process



We have the amplitude

$$i\mathcal{M} = ie^2 \bar{u}(k')\Gamma^\mu(k', k)u(k) \frac{1}{q^2} \bar{u}(p')\gamma_\mu u(p)$$

Here $\Gamma^\mu = \Gamma_1^\mu + \Gamma_2^\mu$ is the proton vertex function defined by

$$\Gamma_1^\mu = \gamma^\mu F_1(q^2) \quad \text{and} \quad \Gamma_2^\mu = \frac{i\sigma^{\mu\nu}q^\nu}{2m} F_2(q^2)$$

To compute the cross section we square the amplitude and average over initial and sum over final spins. We use the properties

$$\gamma^0 (\Gamma_1^\mu)^\dagger \gamma^0 = \Gamma_1^\mu \quad \text{and} \quad \gamma^0 (\Gamma_2^\mu)^\dagger \gamma^0 = -\Gamma_2^\mu$$

This leads to the spin averaged squared amplitude

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{4} \frac{e^4}{q^4} \text{tr}(\not{p}'\gamma_\mu\not{p}\gamma_\nu) \text{tr}((\not{k}' + m)(\Gamma_1^\mu + \Gamma_2^\mu)(\not{k} + m)(\Gamma_1^\nu - \Gamma_2^\nu))$$

We have the symmetric tensor

$$S^{\mu\nu} \equiv \text{tr}(\not{p}'\gamma^\mu\not{p}\gamma^\nu) = 4(p'^\mu p^\nu + p^\mu p'^\nu - \eta^{\mu\nu} p' \cdot p)$$

Thus we need to compute the symmetric parts of the following tensors. First,

$$A^{\mu\nu} \equiv \text{tr}((\not{k}' + m)\Gamma_1^\mu(\not{k} + m)\Gamma_1^\nu) = F_1^2 4(k'^\mu k^\nu + k^\mu k'^\nu + \eta^{\mu\nu}(m^2 - k' \cdot k))$$

Thus,

$$S^{\mu\nu} A_{\mu\nu} = 32 F_1^2 (k' \cdot p' k \cdot p + k' \cdot p k \cdot p' - m^2 p \cdot p')$$

Or, using $q = k' - k = p - p'$,

$$S^{\mu\nu} A_{\mu\nu} = 16 F_1^2 \left(4k \cdot p k \cdot p' + m^2 q^2 \left(1 + \frac{q^2}{2m^2} \right) \right)$$

Secondly,

$$\begin{aligned} B^{\mu\nu} &\equiv \text{tr}((\not{k}' + m)\Gamma_2^\mu(\not{k} + m)\Gamma_1^\nu) - \text{tr}((\not{k}' + m)\Gamma_1^\mu(\not{k} + m)\Gamma_2^\nu) \\ &= m \text{tr}(\not{k}'\Gamma_2^\mu\Gamma_1^\nu) + m \text{tr}(\not{k}\Gamma_1^\nu\Gamma_2^\mu) - m \text{tr}(\not{k}'\Gamma_1^\mu\Gamma_2^\nu) - m \text{tr}(\not{k}\Gamma_2^\nu\Gamma_1^\mu) \end{aligned}$$

Using $k' = k + q$ and symmetrizing,

$$B^{(\mu\nu)} = m \operatorname{tr}(\not{q}\Gamma_2^\mu\Gamma_1^\nu) - m \operatorname{tr}(\not{q}\Gamma_1^\mu\Gamma_2^\nu)$$

Using $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ we find

$$B^{(\mu\nu)} = \frac{1}{4} F_1 F_2 [\operatorname{tr}(\not{q}\gamma^\mu[\gamma^\nu, \not{q}]) - \operatorname{tr}(\not{q}[\gamma^\mu, \not{q}]\gamma^\nu)]$$

Now,

$$[\gamma^\mu, \not{q}] = 2\gamma^\mu \not{q} - 2q^\mu = -2\not{q}\gamma^\mu + 2q^\mu$$

This leads to

$$B^{(\mu\nu)} = 4 F_1 F_2 (q^2 \eta^{\mu\nu} - q^\mu q^\nu)$$

Thus,

$$S^{\mu\nu} B_{\mu\nu} = 64 F_1 F_2 (p \cdot p')^2 = 16 F_1 F_2 q^4$$

Finally we compute

$$C^{\mu\nu} \equiv -\operatorname{tr}((\not{k}' + m)\Gamma_2^\mu(\not{k} + m)\Gamma_2^\nu)$$

We write this as

$$C^{\mu\nu} = -\frac{F_2^2}{16 m^2} \left[\operatorname{tr}(\not{k}'[\gamma^\mu, \not{q}]\not{k}[\gamma^\nu, \not{q}]) + m^2 \operatorname{tr}([\gamma^\mu, \not{q}][\gamma^\nu, \not{q}]) \right]$$

Using the above identities,

$$\operatorname{tr}([\gamma^\mu, \not{q}][\gamma^\nu, \not{q}]) = 2 \operatorname{tr}(\gamma^\mu \not{q}[\gamma^\nu, \not{q}]) = -16 (q^2 \eta^{\mu\nu} - q^\mu q^\nu)$$

Also, using the cyclicity of the trace,

$$\operatorname{tr}(\not{k}'[\gamma^\mu, \not{q}]\not{k}[\gamma^\nu, \not{q}]) = -4 \operatorname{tr}(\not{k}'\gamma^\mu \not{q}\not{k}\not{q}\gamma^\nu) - 4q^\mu q^\nu \operatorname{tr}(\not{k}'\not{k}) + 4q^\mu \operatorname{tr}(\not{k}'\not{k}\not{q}\gamma^\nu) + 4q^\nu \operatorname{tr}(\not{k}\not{k}'\gamma^\mu \not{q})$$

Now,

$$\operatorname{tr}(\not{k}'\gamma^\mu \not{q}\not{k}\not{q}\gamma^\nu) = 2k \cdot q \operatorname{tr}(\not{k}'\gamma^\mu \not{q}\gamma^\nu) - q^2 \operatorname{tr}(\not{k}'\gamma^\mu \not{k}\gamma^\nu)$$

Thus, after some algebra,

$$C^{(\mu\nu)} = \frac{F_2^2}{m^2} \left[(2m^2 - k \cdot q) (q^2 \eta^{\mu\nu} - q^\mu q^\nu) - 2q^2 k^\mu k^\nu + 2k \cdot q (k^\mu q^\nu + q^\mu k^\nu - k \cdot q \eta^{\mu\nu}) \right]$$

After some more algebra,

$$S^{\mu\nu} C_{\mu\nu} = -4F_2^2 \frac{q^2}{m^2} (4k \cdot p k \cdot p' - m^2 q^2)$$

We choose a frame in which the proton is at rest so that $k \cdot p = mE$, $k \cdot p' = mE'$ and $q^2 = -2p \cdot p' = -2EE'(1 - \cos\theta)$. In this frame the differential cross section is given by (see P&S 5.90)

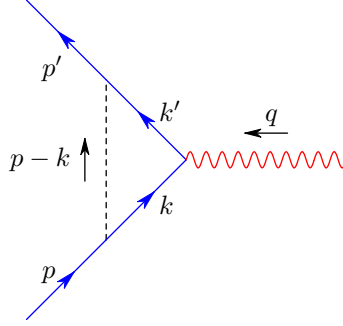
$$\frac{d\sigma}{d \cos \theta} = \frac{1}{32\pi m^2} \left(\frac{E'}{E} \right)^2 \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{\pi\alpha^2}{2E^2} \left(\frac{E'}{E} \right) \frac{S^{\mu\nu} (A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu})}{64m^2 E E' \sin^4(\theta/2)}$$

Where we have used $e^2 = 4\pi\alpha$. Finally, using $E'/E = m/(m + E(1 - \cos\theta))$ and after some algebra,

$$\frac{d\sigma}{d \cos \theta} = \frac{\pi\alpha^2 \left[\left(F_1^2 - \frac{q^2}{4m^2} F_2^2 \right) \cos^2(\theta/2) - \frac{q^2}{2m^2} (F_1 + F_2)^2 \sin^2(\theta/2) \right]}{2E^2 \left[1 + \frac{2E}{m} \sin^2(\theta/2) \right] \sin^4(\theta/2)}$$

2.) Peskin & Schroeder 6.3

We consider the following contribution to the electron-photon vertex function.



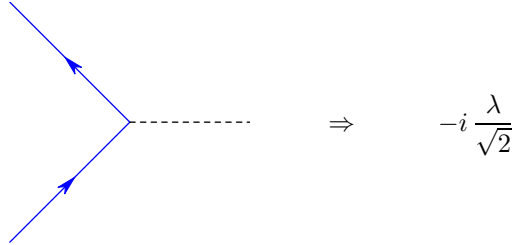
In what follows we will rely heavily on results derived in P&S.

(a)

We consider the interaction Hamiltonian

$$H_{int} = \int d^3x \frac{\lambda}{\sqrt{2}} h \bar{\psi} \psi$$

This leads to the vertex rule



The contribution to the vertex function is

$$\bar{u}(p') \delta \Gamma^\mu(p', p) u(p) = i \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p') (\not{k}' + m) \gamma^\mu (\not{k} + m) u(p)}{((k-p)^2 - m_h^2) (k'^2 - m^2) (k^2 - m^2)}$$

Where we have omitted the $i\epsilon$ term from the propagators. We now introduce Feynman parameters and, following the treatment in P&S, write the integral as

$$\bar{u}(p') \delta \Gamma^\mu(p', p) u(p) = i \frac{\lambda^2}{2} \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2N}{D^3}$$

Where $l \equiv k + yq - zp$, $D \equiv l^2 - \Delta + i\epsilon$ and (see P&S 6.52)

$$\Delta \equiv -xyq^2 + (1-z)^2 m^2 + z m_h^2$$

Also, expressing the numerator in terms of $s \equiv k - l = zp - yq$,

$$N \equiv \bar{u}(p') (\not{l} + \not{s} + \not{q} + m) \gamma^\mu (\not{l} + \not{s} + m) u(p)$$

We now rewrite N as follows

$$N = \bar{u}(p') [A \gamma^\mu + B w^\mu + C q^\mu] u(p)$$

Where $q = p' - p$ and $w \equiv p' + p$. Making use of $\bar{u}(p')\not{q}u(p) = 0$ and $\bar{u}(p')\not{w}u(p) = zm\bar{u}(p')u(p)$ and ignoring terms linear in l which vanish upon integration, after some algebra, we find

$$A = -\frac{1}{2}l^2 - z^2m^2 + (1-x)(x+z-1)q^2$$

$$B = m(z^2 - 1) \quad C = m(z+1)(x-y)$$

See appendix for the B computation. Note that C vanishes upon integration over x and y . We now make use of the Gordon identity

$$2m\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p') [w^\mu + i\sigma^{\mu\nu}q_\nu] u(p)$$

We are interested in the contribution to the anomalous magnetic moment. We write the vertex in terms of the form factors

$$\delta\Gamma^\mu(q, w) = \gamma^\mu (F_1(q^2) - 1) - \frac{w^\mu}{2m} F_2(q^2)$$

Thus

$$F_2(q^2) = -2im\lambda^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{B}{D^3}$$

We evaluate this at $q^2 = 0$ to find

$$\delta a_h = F_2(0) = 2im^2\lambda^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 dz \frac{(1-z^2)(1-z)}{(l^2 - \Delta_0(z) + i\epsilon)^3}$$

Where $\Delta_0(z) = (1-z)^2m^2 + zm_h^2$. Wick rotating and performing the l integral we find

$$\delta a_h = \frac{\lambda^2}{(4\pi)^2} \int_0^1 dz \frac{(1-z^2)(1-z)}{(1-z)^2 + z\rho^2}$$

Where $\rho \equiv m_h/m$. This integral can be evaluated in Mathematica and after some further manipulation we may write

$$(4\pi)^2\lambda^{-2}\delta a_h = \frac{3}{2} - \rho^2 + \frac{1}{2}\rho^2(\rho^2 - 3)\ln(\rho^2) + \frac{\rho^2(4 - 5\rho^2 + \rho^4)}{\sqrt{\rho^4 - 4\rho^2}} \ln\left(\frac{2\rho}{\rho^2 + \sqrt{\rho^4 - 4\rho^2}}\right)$$

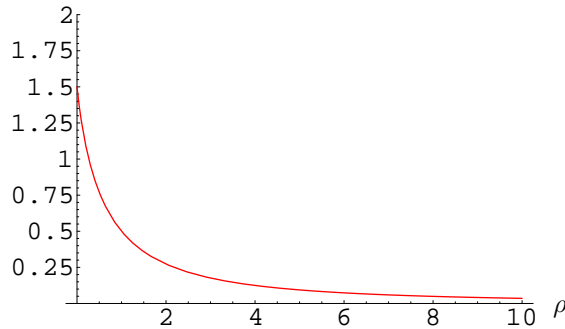
For large ρ we find

$$(4\pi)^2\lambda^{-2}\delta a_h \sim (2\ln\rho - 7/6)\rho^{-2}$$

For small ρ we find

$$(4\pi)^2\lambda^{-2}\delta a_h \sim \frac{3}{2} - \pi\rho - 3\rho^2\ln\rho$$

The following is a graph of $(4\pi)^2\lambda^{-2}\delta a_h$



(b)

As can be seen from the graph above, δa_h is a monotonically decreasing function of ρ which takes the value $\delta a_h = (3/2)\lambda^2/(4\pi)^2$ for $\rho = 0$. For a Higgs particle of mass 60 GeV coupling to the electron $\delta a_h \sim (1.5 \times 10^{-9})\lambda^2/(4\pi)^2$. Thus for $\lambda = 3 \times 10^{-6}$, a massless Higgs leads to $\delta a_h \sim 10^{-13}$ and a 60 GeV Higgs leads to $\delta a_h \sim 10^{-22}$. Both of these bounds are well below $\delta a_h \sim 10^{-10}$ which would require $\lambda = 3.2$ for a 60 GeV Higgs. Of course this value of λ calls into question the perturbative calculation used in deriving it.

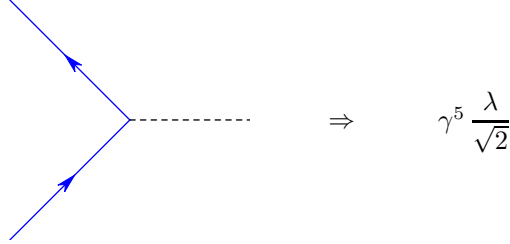
For a Higgs particle of mass 60 GeV coupling to the muon $\delta a_h \sim (3.2 \times 10^{-5})\lambda^2/(4\pi)^2$. Thus for $\lambda = 6 \times 10^{-4}$, a massless Higgs leads to $\delta a_h \sim 10^{-9}$ and a 60 GeV Higgs leads to $\delta a_h \sim 10^{-13}$. Both of these bounds are below $\delta a_h \sim 10^{-8}$ which would require $\lambda = 0.2$ for a 60 GeV Higgs.

(c)

We consider the interaction Hamiltonian

$$H_{int} = \int d^3x \frac{i\lambda}{\sqrt{2}} a \bar{\psi} \gamma^5 \psi$$

This leads to the vertex rule



The contribution to the vertex function is

$$\bar{u}(p') \delta \Gamma^\mu(p', p) u(p) = -i \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p') \gamma^5 (\not{k}' + m) \gamma^\mu (\not{k} + m) \gamma^5 u(p)}{((k-p)^2 - m_h^2)(k^2 - m^2)(k^2 - m^2)}$$

This result is the same as that for the h field above except

$$N = \bar{u}(p') (\not{J} + \not{\not{k}} + \not{k} - m) \gamma^\mu (\not{J} + \not{\not{k}} - m) u(p)$$

After some algebra (see appendix), this leads to

$$B = m(1-z)^2$$

Thus

$$\delta a_a = \frac{\lambda^2}{(4\pi)^2} \int_0^1 dz \frac{(1-z)^3}{(1-z)^2 + z\rho^2}$$

Again we allow Mathematica to do the integral for us and we find

$$(4\pi)^2 \lambda^{-2} \delta a_a = \frac{1}{2} + \rho^2 - \frac{1}{2} \rho^2 (\rho^2 - 1) \ln(\rho^2) + \frac{\rho^4 (3 - \rho^2)}{\sqrt{\rho^4 - 4\rho^2}} \ln\left(\frac{2\rho}{\rho^2 + \sqrt{\rho^4 - 4\rho^2}}\right)$$

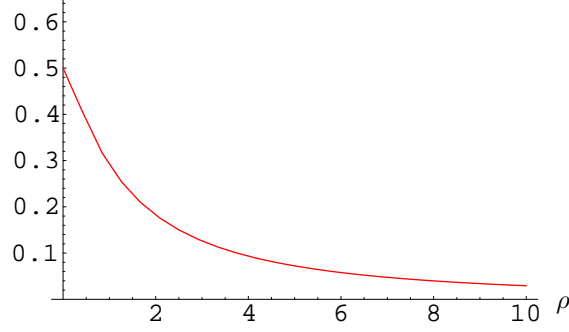
For large ρ we find

$$(4\pi)^2 \lambda^{-2} \delta a_a \sim (2 \ln \rho - 11/6) \rho^{-2}$$

For small ρ we find

$$(4\pi)^2 \lambda^{-2} \delta a_a \sim \frac{1}{2} + (1 + \ln \rho) \rho^2$$

The following is a graph of $(4\pi)^2 \lambda^{-2} \delta a_a$



For an axion that is lighter than an electron coupling to an electron a value of $\delta a_h < 10^{-10}$ would require $\lambda < 1.8 \times 10^{-3}$. For an axion that is lighter than an electron coupling to a muon a value of $\delta a_h < 10^{-8}$ would require $\lambda < 2.3 \times 10^{-4}$.

appendix

We compute the B term contributions to the numerators above. Using $s = zp - yq$, $w = p + p'$ and $q = p' - p$, we systematically ignore terms that will not produce a factor of w^μ

$$\begin{aligned}
N &= \bar{u}(p')(\not{I} + \not{s} + \not{q} \pm m)\gamma^\mu(\not{I} + \not{s} \pm m)u(p) \\
&\Rightarrow \bar{u}(p')[(\not{s} \pm m)\gamma^\mu(\not{s} \pm m) + \not{q}\gamma^\mu(\not{s} \pm m)]u(p) \\
&\Rightarrow \bar{u}(p')[\not{s}\gamma^\mu\not{s} \pm 2ms^\mu + m\not{q}\gamma^\mu(z \pm 1)]u(p) \\
&\Rightarrow \bar{u}(p')[2ms^\mu(z \pm 1) - 2mp^\mu(z \pm 1)]u(p) \\
&\Rightarrow \bar{u}(p')[mw^\mu z(z \pm 1) - mw^\mu(z \pm 1)]u(p)
\end{aligned}$$

Thus

$$B = m(z - 1)(z \pm 1)$$