

QFT 2 : Problem Set 2

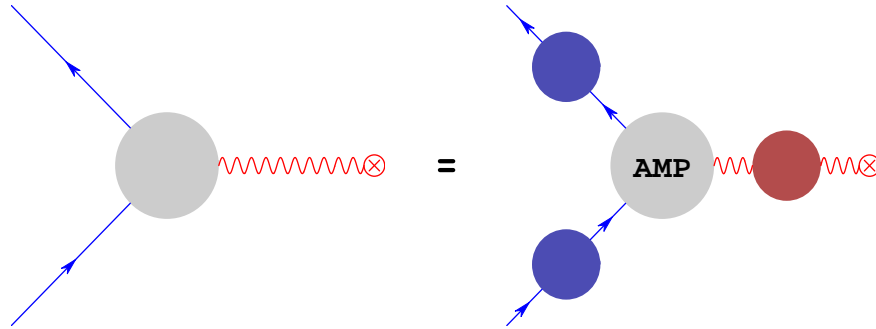
1 Peskin & Schroeder 7.3

We consider a theory of QED coupled to a Yukawa field. The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi \partial_\mu \phi - M^2 \phi^2) + \bar{\psi}(i\cancel{\partial} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e\bar{\psi}A\psi - \frac{\lambda}{\sqrt{2}}\phi\bar{\psi}\psi$$

a)

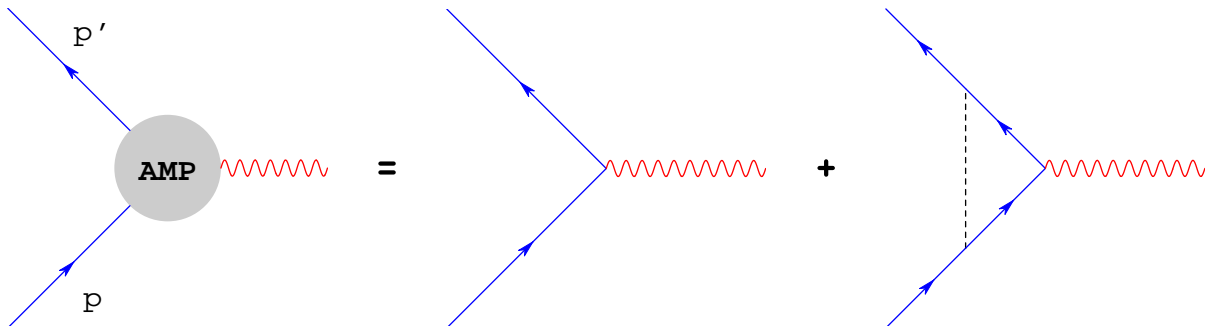
We consider the following process, where on-shell fermions interact with an external photon source:



Where we have:

$$\langle p' | iT[\tilde{A}] | p \rangle = -ie(\sqrt{Z_2})^2 \bar{u}(p')\Gamma^\mu(p', p)u(p) \tilde{A}_\mu(p' - p)$$

Where we have introduced an external potential \tilde{A}_μ and suppressed the spin labels on the incoming and outgoing electron spinors. We consider the finite quantity $Z_2\bar{u}(p')\Gamma^\mu(p', p)u(p)$ to $\mathcal{O}(\lambda^2)$, where $-ie\Gamma^\mu(p', p)$ is given diagrammatically, to $\mathcal{O}(e\lambda^2)$, by the amputated vertex :



Where the Yukawa particles are represented by black dashed lines and we are implicitly ignoring external propagators. Thus we have:

$$\Gamma^\mu(p', p) = \gamma^\mu + \delta\Gamma^\mu(p', p)$$

We also define $Z_1\Gamma^\mu(p, p) = \gamma^\mu$. Our aim is to cancel the divergence in Γ^μ against the divergence in Z_2 , which will be computed below, and verify that $Z_1 = Z_2$ as required by the Ward identity. We first compute $\bar{u}(p')\delta\Gamma^\mu(p', p)u(p)$ using dimensional regularization.

$$\begin{aligned}
& \bar{u}(p')\delta\Gamma^\mu(p', p)u(p) \\
&= \left(\frac{-i\lambda}{\sqrt{2}}\right)^2 \int \frac{d^d k}{(2\pi)^d} \frac{i}{((k-p)^2 - M^2 + i\epsilon)} \bar{u}(p') \frac{i(\not{k}' + m)}{(k'^2 - m^2 + i\epsilon)} \gamma^\mu \frac{i(\not{k} + m)}{(k^2 - m^2 + i\epsilon)} u(p) \\
&= \frac{i}{2} \lambda^2 \int \frac{d^d k}{(2\pi)^d} \frac{\bar{u}(p')(\not{k}' + m)\gamma^\mu(\not{k} + m)u(p)}{((k-p)^2 - M^2 + i\epsilon)(k'^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)}
\end{aligned}$$

Where $k' = k + q$ and $q = p' - p$. We rewrite the denominator using the Feynman parameter trick:

$$\begin{aligned}
\frac{1}{((k-p)^2 - M^2 + i\epsilon)(k'^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} &= \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2}{D^3} \\
D &= x(k^2 - m^2 + i\epsilon) + y(k'^2 - m^2 + i\epsilon) + z((k-p)^2 - M^2 + i\epsilon)
\end{aligned}$$

Using $p^2 = m^2$ and $x + y + z = 1$:

$$D = k^2 + 2k \cdot (yq - zp) + yq^2 + (2z - 1)m^2 - zM^2 + i\epsilon$$

Defining $l = k + (yq - zp)$ and using $2q \cdot p = -q^2$ we have $D = l^2 - \Delta + i\epsilon$ where:

$$\Delta = -xyq^2 + (1-z)^2 m^2 + zM^2$$

Working in the rest frame of the particle with momentum p' , $q^2 = 2m^2 - 2mE_p < 0$, and thus $\Delta > 0$. Thus we have (since $d^d k = d^d l$):

$$\bar{u}(p')\delta\Gamma^\mu(p', p)u(p) = i\lambda^2 \int_0^1 dx dy dz \delta(x+y+z-1) \int \frac{d^d l}{(2\pi)^d} \frac{N}{(l^2 - \Delta + i\epsilon)^3}$$

Where,

$$N = \bar{u}(p')(\not{l} + z\not{p} + (1-y)\not{q} + m)\gamma^\mu(\not{l} + z\not{p} - y\not{q} + m)u(p)$$

Dropping terms linear in l which will vanish upon integration and setting q to zero since our goal is to compute δZ_1 we find:

$$N = \bar{u}(p')\not{l}\gamma^\mu\not{l}u(p) + \bar{u}(p')(z\not{p} + m)\gamma^\mu(z\not{p} + m)u(p)$$

Now, using $\not{p}u(p) = mu(p)$ and $\bar{u}(p')\not{p} = m\bar{u}(p')$,

$$N = \bar{u}(p')\not{l}\gamma^\mu\not{l}u(p) + (1+z)^2 m^2 \bar{u}(p')\gamma^\mu u(p)$$

Anticipating integration, since the denominator is spherically symmetric in l we may make the replacement:

$$\bar{u}(p')(\not{l}\gamma^\mu\not{l})u(p) \rightarrow \frac{1}{d} l^2 \eta_{\alpha\beta} \bar{u}(p')(\gamma^\alpha \gamma^\mu \gamma^\beta)u(p) = l^2 \frac{(2-d)}{d} \bar{u}(p')\gamma^\mu u(p)$$

Thus,

$$N = \bar{u}(p')\gamma^\mu u(p) \left(l^2 \frac{(2-d)}{d} + (1+z)^2 m^2 \right)$$

Using,

$$\int_0^1 dx dy dz \delta(x+y+z-1) f(z) = \int_0^1 dz \int_0^{1-z} dy f(z) = \int_0^1 dz (1-z) f(z)$$

We have,

$$\bar{u}(p')\delta\Gamma^\mu(p', p)u(p) = i\lambda^2 \bar{u}(p')\gamma^\mu u(p) \int_0^1 dz (1-z) \int \frac{d^d l}{(2\pi)^d} \frac{\left(l^2 \frac{(2-d)}{d} + (1+z)^2 m^2 \right)}{(l^2 - \Delta + i\epsilon)^3}$$

We now perform a Wick rotation and define $l^0 = il_E^0$ and $l^i = l_E^i$. Using,

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - \Delta + i\epsilon)^n} = i(-1)^n \int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^n} = \frac{i(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{(n-d/2)}$$

and

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta + i\epsilon)^n} = -i(-1)^n \int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2 + \Delta)^n} = \frac{-i(-1)^n}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - d/2 - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{(n-d/2-1)}$$

We have,

$$\begin{aligned} & \bar{u}(p)\delta\Gamma^\mu(p, p)u(p) \\ &= \frac{-\lambda^2 \bar{u}(p')\gamma^\mu u(p)}{2(4\pi)^2} \int_0^1 dz (1-z) \left(\frac{4\pi}{\Delta}\right)^{(2-d/2)} \left(\Gamma(2-d/2) \frac{(2-d)}{2} - \Gamma(3-d/2) \frac{(1+z)^2 m^2}{\Delta} \right) \end{aligned}$$

We now substitute $\epsilon = (4-d)$, and use $\Gamma(2-d/2) = \Gamma(\epsilon/2) = 2/\epsilon - \gamma + \mathcal{O}(\epsilon)$. Working to $\mathcal{O}(\lambda^2)$ we see from the definition of Z_1 that $\delta Z_1 \gamma^\mu + \delta\Gamma^\mu(p, p) = 0$. Thus, neglecting terms of $\mathcal{O}(\epsilon)$, δZ_1 and is given by:

$$\delta Z_1 = \frac{\lambda^2}{2(4\pi)^2} \int_0^1 dz (1-z) \left(1 + \frac{\epsilon}{2} \log \frac{4\pi}{\Delta} \right) \left(\left(\frac{2}{\epsilon} - \gamma \right) \frac{(\epsilon-2)}{2} - \frac{(1+z)^2 m^2}{\Delta} \right)$$

Or ,

$$\delta Z_1 = \frac{\lambda^2}{2(4\pi)^2} \int_0^1 dz (1-z) \left(1 - \frac{2}{\epsilon} + \gamma - \log \frac{4\pi}{\Delta} - \frac{(1+z)^2 m^2}{\Delta} \right)$$

Substituting $\Delta = (1-z)^2 m^2 + zM^2$,

$$\delta Z_1 = \frac{-\lambda^2}{2(4\pi)^2} \int_0^1 dz (1-z) \left(\frac{2}{\epsilon} - (1+\gamma) + \log \frac{4\pi}{(1-z)^2 m^2 + zM^2} + \frac{(1+z)^2 m^2}{(1-z)^2 m^2 + zM^2} \right)$$

Note that the divergent part is :

$$\delta Z_1 \sim \frac{-\lambda^2}{2\epsilon(4\pi)^2}$$

We now compute Z_2 and show that $Z_2 \bar{u}(p')\Gamma^\mu(p', p)u(p)$ is finite. We anticipate a factor δZ_2 of $\mathcal{O}(\lambda^2)$ such that, to $\mathcal{O}(\lambda^2)$,

$$Z_2 \bar{u}(p')\Gamma^\mu(p', p)u(p) = \bar{u}(p')(\gamma^\mu \delta Z_2 + \Gamma^\mu(p', p))u(p)$$

We see that, from $Z_1 \Gamma^\mu(p, p) = \gamma^\mu$, we must have that $Z_2/Z_1 \simeq (1 + \delta Z_2)(1 - \delta Z_1)$ is finite. Or that the divergent parts are equal $\delta Z_1 \sim \delta Z_2$. In fact, from the Ward identity, $\delta Z_1 = \delta Z_2$. We now compute Z_2 by calculating the Fourier transform of the full electron propagator evaluated at the physical mass. If we define

$$G(p) = \int d^4 x e^{ip \cdot x} \langle \Omega | T(\psi(x) \bar{\psi}(0)) | \Omega \rangle$$

Where ψ and $|\Omega\rangle$ are, respectively, the field operator and vacuum state of the full interacting theory. We then find that,

$$G(p)_{(p^2 \rightarrow m^2)} \sim \frac{iZ_2(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

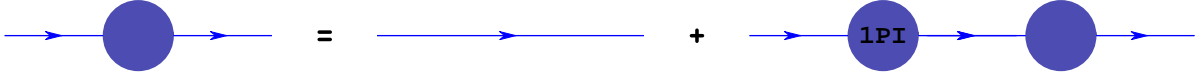
$G(p)$ is defined in a perturbation series as,

$$G(p) = D_F(p) + D_F(p)(-i\Sigma(p))G(p)$$

Here $-i\Sigma(p)$ is the one particle irreducible two point Greens function and $D_F(p)$ is the free Dirac field Feynman propagator :

$$D_F(p) = \frac{i(\not{p} + m_0)}{p^2 - m_0^2 + i\epsilon}$$

We explicitly introduce the bare mass m_0 in the free propagator. It was present above but was neglected since we were working only to $\mathcal{O}(\lambda^2)$. Diagrammatically $G(p)$ is given by,



The solution to the iterative equation for $G(p)$ is given by:

$$G(p) = i(\not{p} - m_0 - \Sigma(\not{p}))^{-1}$$

Expanding $\Sigma(\not{p})$ around the physical mass $m = m_0 + \Sigma(m)$ to $\mathcal{O}(\not{p} - m)$:

$$\Sigma(\not{p}) = \Sigma(m) + (\not{p} - m) \frac{d\Sigma}{d\not{p}}(m)$$

We have, again to $\mathcal{O}(\not{p} - m)$:

$$G(p) = \left(1 - \frac{d\Sigma}{d\not{p}}(m)\right)^{-1} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

This expression has a pole at m , which is why we chose $m = m_0 + \Sigma(m)$. We see that,

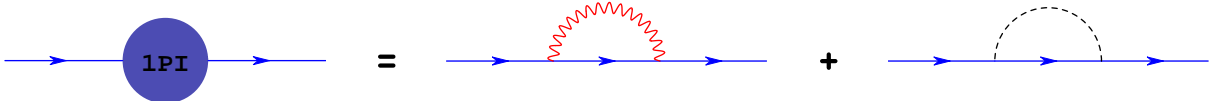
$$Z_2 = \left(1 - \frac{d\Sigma}{d\not{p}}(m)\right)^{-1}$$

We now compute $\Sigma(\not{p})$ to lowest order in both e and λ since we will need the $\mathcal{O}(e^2)$ contribution in part b of the problem.

We express these contributions as,

$$\Sigma_2(\not{p}) = \Sigma_{e^2}(\not{p}) + \Sigma_{\lambda^2}(\not{p})$$

Diagrammatically, $-i\Sigma(\not{p})$ is given to $\mathcal{O}(e^2)$ and $\mathcal{O}(\lambda^2)$ by:



We first compute $\Sigma_{\lambda^2}(\not{p})$:

$$-i\Sigma_{\lambda^2}(\not{p}) = \left(\frac{-i\lambda}{\sqrt{2}}\right)^2 \int \frac{d^d k}{(2\pi)^d} \frac{i}{((p-k)^2 - M^2 + i\epsilon)} \frac{i(\not{k} + m)}{(k^2 - m^2 + i\epsilon)}$$

Introducing a Feynman parameter,

$$\Sigma_{\lambda^2}(\not{p}) = \frac{i\lambda^2}{2} \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{(\not{k} + m)}{(x((p-k)^2 - M^2) + (1-x)(k^2 - m^2) + i\epsilon)^2}$$

Defining $l = k - xp$ we have,

$$\Sigma_{\lambda^2}(\not{p}) = \frac{i\lambda^2}{2} \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{(\not{l} + x\not{p} + m)}{(l^2 + x(1-x)p^2 - xM^2 - (1-x)m^2 + i\epsilon)^2}$$

Ignoring terms linear in l , defining $\Delta = -x(1-x)p^2 + xM^2 + (1-x)m^2$ and Wick rotating as above :

$$\Sigma_{\lambda^2}(\not{p}) = \frac{-\lambda^2}{2} \int_0^1 dx \int \frac{d^d l_E}{(2\pi)^d} \frac{(x\not{p} + m)}{(l_E^2 + \Delta)^2}$$

Thus, using relations from above,

$$\Sigma_{\lambda^2}(\not{p}) = \frac{-\lambda^2}{2(4\pi)^2} \int_0^1 dx (x\not{p} + m) \left(\frac{2}{\epsilon} - \gamma\right) \left(1 + \frac{\epsilon}{2} \log \frac{4\pi}{\Delta}\right)$$

Thus,

$$\Sigma_{\lambda^2}(\not{p}) = \frac{-\lambda^2}{2(4\pi)^2} \int_0^1 dx (x\not{p} + m) \left(\frac{2}{\epsilon} - \gamma + \log \frac{4\pi}{xM^2 + (1-x)m^2 - x(1-x)p^2}\right)$$

We have to $\mathcal{O}(\lambda^2)$:

$$\begin{aligned} \delta Z_2 &= \left(\left(1 - \frac{d\Sigma_{\lambda^2}}{d\not{p}}(m)\right)^{-1} - 1 \right) \simeq \frac{d\Sigma_{\lambda^2}}{d\not{p}}(m) \\ \delta Z_2 &= \frac{-\lambda^2}{2(4\pi)^2} \int_0^1 dx x \left(\frac{2}{\epsilon} - \gamma + \log \frac{4\pi}{xM^2 + (1-x)^2 m^2} + \frac{2(1-x^2)m^2}{xM^2 + (1-x)^2 m^2} \right) \end{aligned}$$

Changing variables to $z = 1 - x$:

$$\delta Z_2 = \frac{-\lambda^2}{2(4\pi)^2} \int_0^1 dz (1-z) \left(\frac{2}{\epsilon} - \gamma + \log \frac{4\pi}{(1-z)M^2 + z^2 m^2} + \frac{2z(2-z)m^2}{(1-z)M^2 + z^2 m^2} \right)$$

Note that the divergent part is :

$$\delta Z_2 \sim \frac{-\lambda^2}{2\epsilon(4\pi)^2}$$

This is the same result found above for the divergent part of δZ_1 . From above,

$$\delta Z_1 = \frac{-\lambda^2}{2(4\pi)^2} \int_0^1 dz (1-z) \left(\frac{2}{\epsilon} - (1+\gamma) + \log \frac{4\pi}{(1-z)^2 m^2 + zM^2} + \frac{(1+z)^2 m^2}{(1-z)^2 m^2 + zM^2} \right)$$

Thus,

$$\begin{aligned} \delta Z_1 - \delta Z_2 &= \frac{\lambda^2}{2(4\pi)^2} \int_0^1 dz (1-z) \left(1 + \log \left((1-z)^2 m^2 + zM^2 \right) - \frac{(1+z)^2 m^2}{(1-z)^2 m^2 + zM^2} \right) \\ &\quad + \frac{\lambda^2}{2(4\pi)^2} \int_0^1 dz (1-z) \left(-\log \left((1-z)M^2 + z^2 m^2 \right) + \frac{2z(2-z)m^2}{(1-z)M^2 + z^2 m^2} \right) \end{aligned}$$

Or,

$$\delta Z_1 - \delta Z_2 = \frac{\lambda^2}{2(4\pi)^2} \int_0^1 dz \left((1-z) + (1-2z) \log(f(z)) + \frac{2z(1-z^2)m^2 - (1+z)^2m^2}{f(z)} \right)$$

Where,

$$f(z) \equiv (1-z)^2m^2 + zM^2$$

Now,

$$\frac{d}{dz} ((z-z^2) \log(f(z))) = (1-2z) \log(f(z)) + \frac{(z-z^2)(-2(1-z)m^2 + M^2)}{f(z)}$$

After a little algebra we find that $\delta Z_1 = \delta Z_2$. Thus the Ward identity is satisfied. This is generally true for this theory as a consequence of the gauge invariance of the Lagrangian.

Since we will need the $\mathcal{O}(e^2)$ contribution to δZ_2 in part b, we now compute $\Sigma_{e^2}(\not{p})$ using dimensional regularization :

$$-i\Sigma_{e^2}(\not{p}) = (-ie)^2 \int \frac{d^d k}{(2\pi)^d} \frac{-i}{((p-k)^2 + i\epsilon)} \frac{i\gamma^\mu(\not{k} + m)\gamma_\mu}{(k^2 - m^2 + i\epsilon)}$$

Introducing a Feynman parameter,

$$-i\Sigma_{e^2}(\not{p}) = -e^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu(\not{k} + m)\gamma_\mu}{(x(p-k)^2 + (1-x)(k^2 - m^2) + i\epsilon)^2}$$

Defining $l = k - xp$ we have and ignoring terms linear in l ,

$$-i\Sigma_{e^2}(\not{p}) = -e^2 \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{\gamma^\mu(x\not{p} + m)\gamma_\mu}{(l^2 - \Delta + i\epsilon)^2}$$

Where, $\Delta = -x(1-x)p^2 + (1-x)m^2$. Using $\gamma^\mu(x\not{p} + m)\gamma_\mu = (2-d)x\not{p} + dm$ and Wick rotating we have :

$$-i\Sigma_{e^2}(\not{p}) = -ie^2 \int_0^1 dx \int \frac{d^d l_E}{(2\pi)^d} \frac{(2-d)x\not{p} + dm}{(l_E^2 + \Delta)^2}$$

Using relations defined above,

$$-i\Sigma_{e^2}(\not{p}) = \frac{-ie^2}{(4\pi)^2} \int_0^1 dx \left(\frac{2}{\epsilon} - \gamma + \log \frac{4\pi}{\Delta} \right) ((\epsilon-2)x\not{p} + (4-\epsilon)m)$$

$$-i\Sigma_{e^2}(\not{p}) = \frac{ie^2}{(4\pi)^2} \int_0^1 dx \left((2x\not{p} - 4m) \left(\frac{2}{\epsilon} - 1 - \gamma + \log \frac{4\pi}{\Delta} \right) - 2m \right)$$

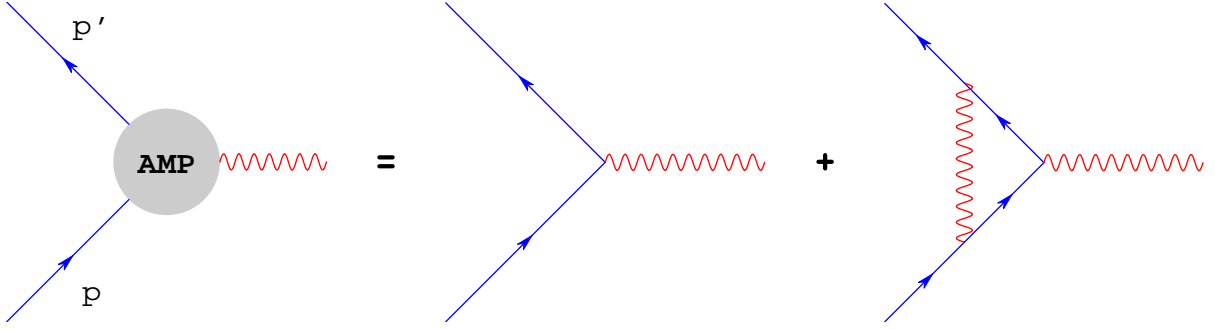
To $\mathcal{O}(e^2)$:

$$\delta Z_2 = \frac{-2e^2}{(4\pi)^2} \int_0^1 dx (x) \left(\frac{2}{\epsilon} - 1 - \gamma + \log \frac{4\pi}{(1-x)^2m^2} + \frac{2(x-2)(1-x)}{(1-x)^2} \right)$$

Note that the divergent part is :

$$\delta Z_2 \sim \frac{-2e^2}{\epsilon(4\pi)^2}$$

We confirm that (to $\mathcal{O}(e^2)$) $\delta Z_1 = \delta Z_2$. The contribution to Z_1 comes from the amputated vertex diagram:



We compute δZ_1 from the definition $\delta Z_1 \gamma^\mu + \delta \Gamma^\mu(p, p) = 0$. We compute $\bar{u}(p') \delta \Gamma^\mu(p', p) u(p)$ to $\mathcal{O}(e^2)$ using dimensional regularization.

$$\begin{aligned} & \bar{u}(p') \delta \Gamma^\mu(p', p) u(p) \\ &= -ie^2 \int \frac{d^d k}{(2\pi)^d} \frac{\bar{u}(p') \gamma^\nu (\not{k}' + m) \gamma^\mu (\not{k} + m) \gamma_\nu u(p)}{((k-p)^2 + i\epsilon)(k'^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} \end{aligned}$$

The denominator can be rewritten as,

$$\frac{1}{((k-p)^2 + i\epsilon)(k'^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} = \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2}{D^3}$$

Where, using $k' = k + q$ and $q = p' - p$,

$$D = k^2 + 2k \cdot (yq - zp) + yq^2 + (2z-1)m^2 + i\epsilon$$

Defining $l = k + (yq - zp)$, $D = l^2 - \Delta + i\epsilon$. Where,

$$\Delta = -xyq^2 + (1-z)^2 m^2$$

We may thus write,

$$\bar{u}(p') \delta \Gamma^\mu(p', p) u(p) = -2ie^2 \int_0^1 dx dy dz \delta(x+y+z-1) \int \frac{d^d l}{(2\pi)^d} \frac{N}{(l^2 - \Delta + i\epsilon)^3}$$

Where,

$$N = \bar{u}(p') \gamma^\nu (\not{l} + z\not{p} + (1-y)\not{q} + m) \gamma^\mu (\not{l} + z\not{p} - y\not{q} + m) \gamma_\nu u(p)$$

Dropping terms linear in l which will vanish upon integration and setting q to zero since our goal is to compute δZ_1 :

$$N = \bar{u}(p) (\gamma^\nu \not{l} \gamma^\mu \not{l} \gamma_\nu + \gamma^\nu (z\not{p} + m) \gamma^\mu (z\not{p} + m) \gamma_\nu) u(p)$$

Now,

$$\gamma^\nu \not{l} \gamma^\mu \not{l} \gamma_\nu = l_\alpha l_\beta \gamma^\nu \gamma^\alpha \gamma^\mu \gamma^\beta \gamma_\nu$$

Anticipating integration we replace $l_\alpha l_\beta$ by $\frac{1}{d} l^2 \eta_{\alpha\beta}$. Thus, using $\gamma^\alpha \gamma^\mu \gamma_\alpha = (2-d)\gamma^\mu$ we have:

$$\gamma^\nu \not{l} \gamma^\mu \not{l} \gamma_\nu = \frac{1}{d} l^2 \gamma^\nu \gamma^\alpha \gamma^\mu \gamma_\alpha \gamma_\nu = \frac{(2-d)^2}{d} l^2 \gamma^\mu$$

Now,

$$(z\not{p} + m) \gamma_\nu u(p) = \gamma_\nu m(1-z)u(p) + 2zp_\nu u(p)$$

$$\bar{u}(p)\gamma^\nu(z\not{p} + m) = m(1-z)\bar{u}(p)\gamma^\nu + 2zp^\nu\bar{u}(p)$$

Thus,

$$\begin{aligned}\bar{u}(p)\gamma^\nu(z\not{p} + m)\gamma^\mu(z\not{p} + m)\gamma_\nu u(p) &= \bar{u}(p)(m(1-z)\gamma^\nu + 2zp^\nu)\gamma^\mu(\gamma_\nu m(1-z) + 2zp_\nu)u(p) \\ &= m^2\bar{u}(p)\gamma^\mu u(p)((1-z)^2(2-d) + 4z)\end{aligned}$$

Thus,

$$N = \bar{u}(p)\gamma^\mu u(p) \left(\frac{(2-d)^2}{d}l^2 + m^2((1-z)^2(2-d) + 4z) \right)$$

To $\mathcal{O}(e^2)$:

$$\delta Z_1 = 2ie^2 \int_0^1 dz (1-z) \int \frac{d^d l}{(2\pi)^d} \frac{\left(\frac{(2-d)^2}{d}l^2 + m^2((1-z)^2(2-d) + 4z) \right)}{(l^2 - \Delta + i\epsilon)^3}$$

Using relations defined above:

$$\delta Z_1 = \frac{-e^2}{(4\pi)^2} \int_0^1 dz (1-z) \left(\frac{4\pi}{\Delta} \right)^{(2-d/2)} \left(\Gamma(2-d/2) \frac{(2-d)^2}{2} - \frac{m^2}{\Delta} ((1-z)^2(2-d) + 4z) \right)$$

Working to $\mathcal{O}(\epsilon = (4-d))$:

$$\delta Z_1 = \frac{-e^2}{(4\pi)^2} \int_0^1 dz (1-z) \left(1 + \frac{\epsilon}{2} \log \frac{4\pi}{\Delta} \right) \left(\left(\frac{2}{\epsilon} - \gamma \right) \frac{(\epsilon-2)^2}{2} + \frac{m^2}{\Delta} (2(1-z)^2 - 4z) \right)$$

Or,

$$\delta Z_1 = \frac{-2e^2}{(4\pi)^2} \int_0^1 dz (1-z) \left(\frac{2}{\epsilon} - 2 - \gamma + \log \frac{4\pi}{\Delta} + \frac{m^2}{\Delta} ((1-z)^2 - 2z) \right)$$

Note that the divergent part is :

$$\delta Z_1 \sim \frac{-2e^2}{\epsilon(4\pi)^2}$$

This is the same result computed above for δZ_2 . Setting $\Delta = (1-z)^2 m^2$:

$$\delta Z_1 = \frac{-2e^2}{(4\pi)^2} \int_0^1 dz (1-z) \left(\frac{2}{\epsilon} - 2 - \gamma + \log \frac{4\pi}{(1-z)^2 m^2} + \frac{((1-z)^2 - 2z)}{(1-z)^2} \right)$$

Thus, to $\mathcal{O}(e^2)$:

$$\begin{aligned}\delta Z_1 - \delta Z_2 &= \frac{-2e^2}{(4\pi)^2} \int_0^1 dz (1-z) \left(\frac{2}{\epsilon} - 2 - \gamma + \log \frac{4\pi}{(1-z)^2 m^2} + \frac{((1-z)^2 - 2z)}{(1-z)^2} \right) \\ &\quad + \frac{2e^2}{(4\pi)^2} \int_0^1 dz (1-z) \left(\frac{2}{\epsilon} - 2 - \gamma + \log \frac{4\pi}{z^2 m^2} - \frac{2(z+1)z - z^2}{z^2} \right) \\ \delta Z_1 - \delta Z_2 &= \frac{-2e^2}{(4\pi)^2} \int_0^1 dz (1-z) \left(\log \frac{z^2}{(1-z)^2} + \frac{2(z+1)z}{z^2} - \frac{2z}{(1-z)^2} \right)\end{aligned}$$

Now,

$$\int_0^1 dz (1-z) \log \frac{z^2}{(1-z)^2} = \int_0^1 dz 2((1-z) \log z - z \log z)$$

From, $\frac{d}{dz}(z^n \log z) = z^{(n-1)}(1 + n \log z)$,

$$\int_0^1 dz 2(\log z - 2z \log z) = 2 \int_0^1 dz (-1 + z) = -1$$

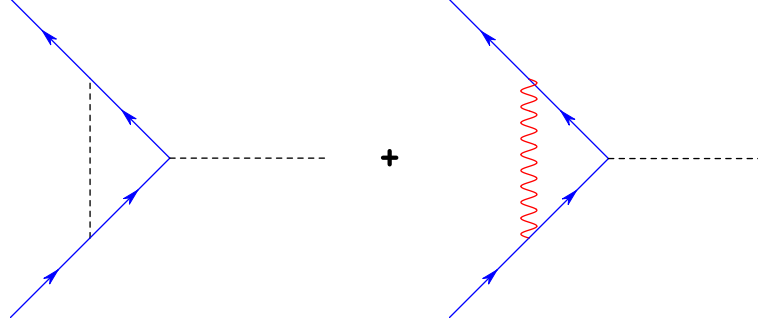
And,

$$\int_0^1 dz 2z(1-z) \left(\frac{(z+1)}{z^2} - \frac{1}{(1-z)^2} \right) = \int_0^1 dz 2z(1-z) \left(\frac{(z+1)}{z^2} - \frac{1}{z^2} \right) = 1$$

Thus $\delta Z_1 = \delta Z_2$ as is required by the Ward identity.

b)

We now consider the $\phi\bar{\psi}\psi$ vertex and the finite quantity $Z_2\bar{u}(p')\hat{\Gamma}(p', p)u(p)$ to $\mathcal{O}(\lambda^2, e^2)$, where $\hat{\Gamma}(p', p) = 1 + \delta\hat{\Gamma}(p', p)$ and $\hat{Z}_1\hat{\Gamma}(p, p) = 1$. $-i\frac{\lambda}{\sqrt{2}}\delta\hat{\Gamma}(p', p)$ is given diagrammatically, to $\mathcal{O}(\lambda^3, \lambda e^2)$, by the amputated vertex graphs :



We compute the divergent parts each of these graphs and attempt to cancel the divergences against those arising in the respective propagators computed above. That is we attempt to cancel the $\mathcal{O}(\lambda^2, e^2)$ divergences in \hat{Z}_1 against those in Z_2 . To $\mathcal{O}(\lambda^2, e^2)$ we have $\delta\hat{\Gamma}(p', p) = \delta\hat{\Gamma}_{\lambda^2}(p', p) + \delta\hat{\Gamma}_{e^2}(p', p)$.

$$\begin{aligned} & \bar{u}(p')\delta\hat{\Gamma}_{\lambda^2}(p', p)u(p) \\ &= \left(\frac{-i\lambda}{\sqrt{2}}\right)^2 \int \frac{d^d k}{(2\pi)^d} \frac{i}{((k-p)^2 - M^2 + i\epsilon)} \bar{u}(p') \frac{i(\not{k}' + m)}{(k'^2 - m^2 + i\epsilon)} \frac{i(\not{k} + m)}{(k^2 - m^2 + i\epsilon)} u(p) \end{aligned}$$

Where $k' = k + q$ and $q = p' - p$. We again rewrite the denominator using the Feynman parameter trick. We also set $q = 0$ from the outset of the calculation. Freely using results derived above:

$$\begin{aligned} & \bar{u}(p)\delta\hat{\Gamma}_{\lambda^2}(p, p)u(p) \\ &= i\lambda^2 \int_0^1 dz (1-z) \int \frac{d^d l}{(2\pi)^d} \frac{\bar{u}(p)(\not{l} + z\not{p} + m)(\not{l} + z\not{p} + m)u(p)}{(l^2 - \Delta + i\epsilon)^3} \end{aligned}$$

Where $l = k - zp$, $\Delta = (1-z)^2 m^2 + zM^2$. Ignoring terms linear in l we have:

$$\bar{u}(p)\delta\hat{\Gamma}_{\lambda^2}(p, p)u(p) = i\lambda^2 \int_0^1 dz (1-z) \int \frac{d^d l}{(2\pi)^d} \frac{\bar{u}(p)(l^2 + (1+z)^2 m^2)u(p)}{(l^2 - \Delta + i\epsilon)^3}$$

Thus, to $\mathcal{O}(\lambda^2)$,

$$\delta\hat{Z}_1 = -i\lambda^2 \int_0^1 dz (1-z) \int \frac{d^d l}{(2\pi)^d} \frac{(l^2 + (1+z)^2 m^2)}{(l^2 - \Delta + i\epsilon)^3}$$

Or,

$$\delta\hat{Z}_1 = \frac{\lambda^2}{2(4\pi)^2} \int_0^1 dz (1-z) \left(\frac{4\pi}{\Delta}\right)^{(2-d/2)} \left(\frac{d}{2}\Gamma(2-d/2) - \frac{(1+z)^2 m^2}{\Delta}\right)$$

Working to $\mathcal{O}(\epsilon = (4-d))$:

$$\delta\hat{Z}_1 = \frac{\lambda^2}{2(4\pi)^2} \int_0^1 dz (1-z) \left(1 + \frac{\epsilon}{2} \log \frac{4\pi}{\Delta}\right) \left(\frac{(4-\epsilon)}{2} \left(\frac{2}{\epsilon} - \gamma\right) - \frac{(1+z)^2 m^2}{\Delta}\right)$$

$$\delta\widehat{Z}_1 = \frac{\lambda^2}{2(4\pi)^2} \int_0^1 dz (1-z) \left(\frac{4}{\epsilon} - 2\gamma - 1 + 2 \log \frac{4\pi}{\Delta} - \frac{(1+z)^2 m^2}{\Delta} \right)$$

Thus the divergent part of $\delta\widehat{Z}_1$ (to $\mathcal{O}(\lambda^2)$) is :

$$\delta\widehat{Z}_1 \sim \frac{-\lambda^2}{\epsilon(4\pi)^2}$$

This does not equal the divergent part of the $\mathcal{O}(\lambda^2)$ contribution to δZ_2 derived above :

$$\delta Z_2 \sim \frac{-\lambda^2}{2\epsilon(4\pi)^2}$$

We now compute the divergent part of $\delta\widehat{Z}_1$ to $\mathcal{O}(e^2)$:

$$\begin{aligned} & \bar{u}(p') \delta\widehat{\Gamma}_{e^2}(p', p) u(p) \\ &= (-ie)^2 \int \frac{d^d k}{(2\pi)^d} \frac{-i}{((k-p)^2 + i\epsilon)} \bar{u}(p') \gamma^\mu \frac{i(\not{k} + m)}{(k'^2 - m^2 + i\epsilon)} \frac{i(\not{k} + m)}{(k^2 - m^2 + i\epsilon)} \gamma_\mu u(p) \end{aligned}$$

Again using the Feynman parameter trick and setting $q = 0$:

$$\begin{aligned} & \bar{u}(p') \delta\widehat{\Gamma}_{e^2}(p', p) u(p) \\ &= -2ie^2 \int_0^1 dz (1-z) \int \frac{d^d l}{(2\pi)^d} \frac{\bar{u}(p) \gamma^\mu (\not{l} + z\not{p} + m) (\not{l} + z\not{p} + m) \gamma_\mu u(p)}{(l^2 - \Delta + i\epsilon)^3} \end{aligned}$$

Where $l = k - zp$, $\Delta = (1-z)^2 m^2$. Ignoring terms linear in l we have :

$$\bar{u}(p') \delta\widehat{\Gamma}_{e^2}(p', p) u(p) = -2ie^2 \int_0^1 dz (1-z) \int \frac{d^d l}{(2\pi)^d} \frac{\bar{u}(p) (dl^2 + d(1+z)^2 m^2) u(p)}{(l^2 - \Delta + i\epsilon)^3}$$

Thus, to $\mathcal{O}(e^2)$,

$$\delta\widehat{Z}_1 = 2ie^2 \int_0^1 dz (1-z) \int \frac{d^d l}{(2\pi)^d} \frac{(dl^2 + d(1+z)^2 m^2)}{(l^2 - \Delta + i\epsilon)^3}$$

Or,

$$\delta\widehat{Z}_1 = \frac{-e^2}{(4\pi)^2} \int_0^1 dz (1-z) d \left(\frac{4\pi}{\Delta} \right)^{(2-d/2)} \left(\frac{d}{2} \Gamma(2-d/2) - \frac{(1+z)^2 m^2}{\Delta} \right)$$

Working to $\mathcal{O}(\epsilon = (4-d))$:

$$\delta\widehat{Z}_1 = \frac{-e^2}{(4\pi)^2} \int_0^1 dz (1-z) (4-\epsilon) \left(1 + \frac{\epsilon}{2} \log \frac{4\pi}{\Delta} \right) \left(\frac{(4-\epsilon)}{2} \left(\frac{2}{\epsilon} - \gamma \right) - \frac{(1+z)^2 m^2}{\Delta} \right)$$

Using the preceding similar $\mathcal{O}(\lambda^2)$ result ,

$$\delta\widehat{Z}_1 = \frac{-4e^2}{(4\pi)^2} \int_0^1 dz (1-z) \left(\frac{4}{\epsilon} - 2\gamma - 2 + 2 \log \frac{4\pi}{\Delta} - \frac{(1+z)^2 m^2}{\Delta} \right)$$

Thus the divergent part of $\delta\widehat{Z}_1$ (to $\mathcal{O}(e^2)$) is :

$$\delta\widehat{Z}_1 \sim \frac{-8e^2}{\epsilon(4\pi)^2}$$

This does not equal the divergent part of the $\mathcal{O}(e^2)$ contribution to δZ_2 derived above :

$$\delta Z_2 \sim \frac{-2e^2}{\epsilon(4\pi)^2}$$