QFT 2 : Problem Set 3

1 Peskin & Schroeder 10.2

We consider the pseudoscalar Yukawa Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \phi\, \partial_\mu \phi - M^2\phi^2\right) + \overline{\psi}\,(i\partial\!\!\!/ - m)\,\psi - i\,g\,\phi\,\overline{\psi}\gamma^5\psi$$

Note that the action is invariant under a parity transformation:

$$\mathcal{P}\psi(t,\mathbf{x})\mathcal{P}^\dagger = \gamma^0\psi(t,-\mathbf{x})$$
$$\mathcal{P}\phi(t,\mathbf{x})\mathcal{P}^\dagger = -\phi(t,-\mathbf{x})$$

a)

From simple power counting we see that ψ and ϕ have mass dimension 3/2 and 1 respectively. We denote this as $[\psi] = 3/2$ and $[\phi] = 1$. Since this requires $[g] = 0$, we see that this theory is renormalizable. We introduce the following notation to investigate the superficial degree of divergence of Feynman diagrams:

N_F = # of external fermion lines
N_B = # of external boson lines
P_F = # of fermion propagators
P_B = # of boson propagators
V_F = # of 2-fermion/1-boson vertices
L = # of loops

We compute the superficial degree of divergence $D = 4L - P_F - 2P_B$ in $d = 4$ dimensions . We have,

$$L = P_F + P_B - V_F + 1$$

$$2V_F = 2P_F + N_F$$

$$V_F = 2P_B + N_B$$

Thus,

$$D = 3P_F + 2P_B - 4V_F + 4$$

$$D = 4 - \frac{3}{2}N_F - N_B$$

We see that there is a superficially divergent $(N_F, N_B) = (0, 4)$ $(D = 0)$ vertex which suggests we must include a scalar self-interaction if the theory is to be renormalizable. We consider the new Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \phi_b \partial_\mu \phi_b - M_b^2\phi_b^2\right) + \overline{\psi}_b\,(i\partial\!\!\!/ - m_b)\,\psi_b - ig_b\phi_b\overline{\psi}_b\gamma^5\psi_b - \frac{\lambda_b}{4!}\phi_b^4$$

The b subscricpt denotes bare quantities. We work in d dimensionsand repeat the power counting arguments given above. Here we have $[\psi] = (d-1)/2$, $[\phi] = (d-2)/2$, $[\lambda] = (4-d)$ and $[g] = (4-d)/2$. We introduce,

V_B = # of 4-boson vertices

We compute the superficial degree of divergence $D = dL - P_F - 2P_B$ in d dimensions . Thus, we have,

$$L = P_F + P_B - V_F - V_B + 1$$

$$2V_F = 2P_F + N_F$$

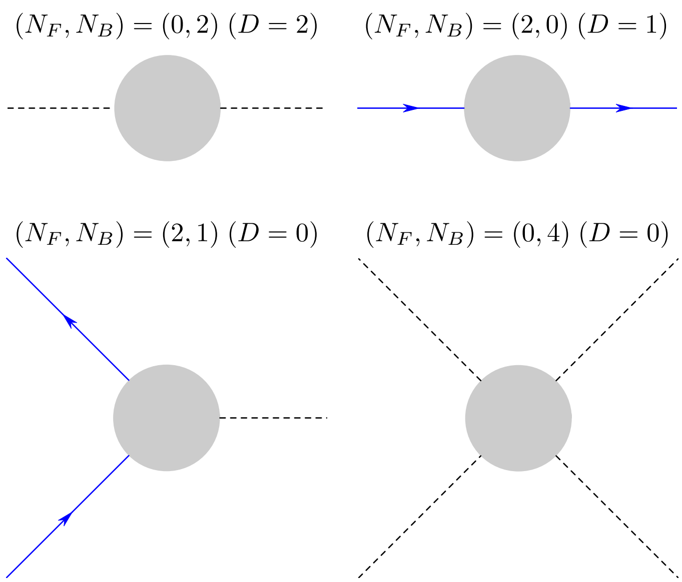
$$V_F + 4V_B = 2P_B + N_B$$

Thus,

$$D = d(P_F + P_B - V_F - V_B + 1) - P_F - 2P_B$$

$$D = d - N_F((d-1)/2) - N_B((d-2)/2) + V_F((d-4)/2) + V_B(d-4)$$

Of course, in $d = 4$ dimensions, $D = 4 - \frac{3}{2}N_F - N_B$ and we find the same superficially divergent amplitudes as considered above. We ignore the $(N_F, N_B) = (0,0)$ $(D = 4)$ vacuum bubbles which do not contribute to any S-matrix element. We also ignore the fermion one-point function $(N_F, N_B) = (1,0)$ $(D = 5/2)$ since this fails to respect Lorentz and charge conjugation invariance. The same argument applies to the $(N_F, N_B) = (1,1)$ $(D = 3/2)$ and $(N_F, N_B) = (1,2)$ $(D = 1/2)$ diagrams. We ignore the boson one-point $(N_F, N_B) = (0,1)$ $(D = 3)$ and three-point $(N_F, N_B) = (0,3)$ $(D = 1)$ functions since these fail to respect invariance under parity transformations. We enumerate the remaining superficially divergent amplitudes diagrammatically:



We now reexpress the bare variables in terms of renormalized variables as follows:

$$\phi_b = \sqrt{Z_\phi}\,\phi \qquad \psi_b = \sqrt{Z_\psi}\,\psi$$

$$Z_\phi M_b^2 = M^2 + \delta_M \qquad Z_\psi m_b = m + \delta_m$$

$$Z_\phi^2 \lambda_b = \lambda + \delta_\lambda \qquad g_b Z_\psi \sqrt{Z_\phi} = g(1 + \delta_1)$$

Defining $\delta_\phi = Z_\phi - 1$ and $\delta_\psi = Z_\psi - 1$, the Lagrangian takes the form:

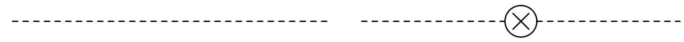
$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi\partial_\mu\phi - M^2\phi^2) + \overline{\psi}(i\partial\!\!\!/ - m)\psi - ig\phi\overline{\psi}\gamma^5\psi - \frac{\lambda}{4!}\phi^4$$

$$+\frac{1}{2}(\delta_\phi\partial^\mu\phi\partial_\mu\phi - \delta_M\phi^2) + \overline{\psi}(\delta_\psi i\partial\!\!\!/ - \delta_m)\psi - ig\delta_1\phi\overline{\psi}\gamma^5\psi - \frac{\delta_\lambda}{4!}\phi^4$$

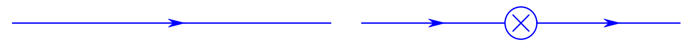
Note that δ_ϕ , δ_ψ , δ_M and δ_m can have contributions to first order in g and λ but δ_λ must be higher than $\mathcal{O}(\lambda)$ and δ_g must be higher than $\mathcal{O}(g)$.

We have the following Feynman rules:

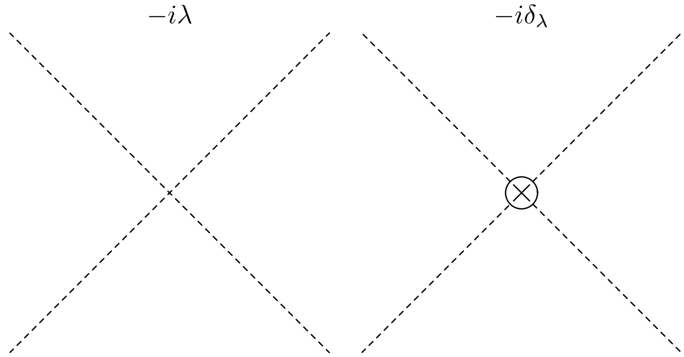
$i\left(p^2 - M^2 + i\epsilon\right)^{-1}$ $\qquad$ $i\left(p^2\delta_\phi - \delta_M\right)$



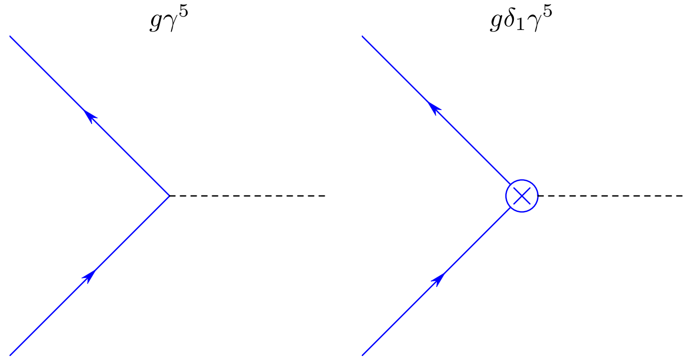
$i\left(\not{p} - m + i\epsilon\right)^{-1}$ $\qquad$ $i\left(\not{p}\delta_\psi - \delta_m\right)$



$-i\lambda$ $\qquad$ $-i\delta_\lambda$



$g\gamma^5$ $\qquad$ $g\delta_1\gamma^5$

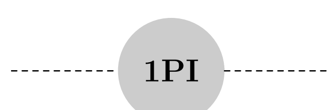


b)

We now define the renormalization conditions used to fix the values of the counterterm coefficients. We begin with the pseudo-scalar propagator given above. We have,

$i\left(p^2 - M^2 - \mathcal{M}^2(p^2)\right)^{-1} =$

$-i\mathcal{M}^2(p^2) =$



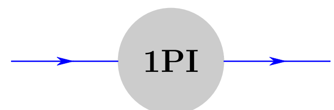
We choose the normalization conditions:

$$\mathcal{M}^2(p^2)\,|_{p^2=M^2}\; = 0 \quad \text{and} \quad \frac{d}{dp^2}\mathcal{M}^2(p^2)\,|_{p^2=M^2}\; = 0$$

We now define the renormalization conditions associated with the fermion propagator. We have,

$$i\,(\not{p} - m - \Sigma(\not{p}))^{-1} \; =$$

$$-i\Sigma(\not{p}) =$$

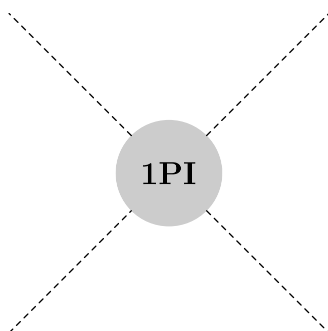


We choose the normalization conditions:

$$\Sigma(\not{p})\,|_{\not{p}=m}\; = 0 \quad \text{and} \quad \frac{d}{d\not{p}}\Sigma(\not{p})\,|_{\not{p}=m}\; = 0$$

The fifth condition applies to the vertex involving 4 pseudo-scalars. If we define

$$-i\mathcal{A} \; =$$

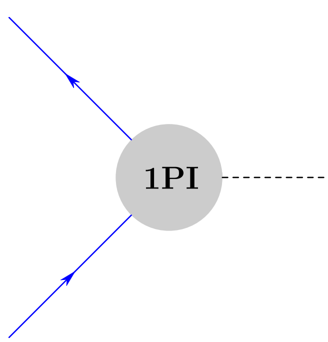


We may choose the normalization condition, in terms of the Mandelstam variables (s, t, u) as follows:

$$-i\mathcal{A} = -i\lambda \quad \text{at} \quad s^2 = 4M^2\;,\; t = u = 0$$

The sixth and final condition applies to the vertex involving 1 pseudo-scalar and 2 fermions. If we define

$$g\,\Gamma \; =$$

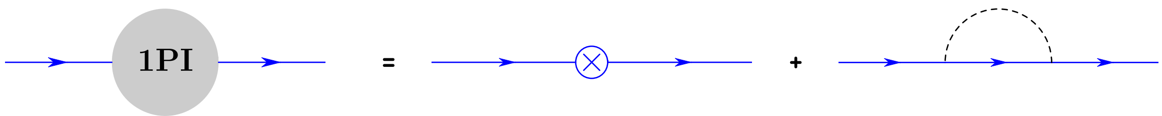


We may choose the normalization condition to reflect the physical coupling at zero momentum transfer:

$$g\Gamma = g\gamma^5 \quad \text{at} \quad p' - p = q = 0$$

Here p and p' are the incoming and outgoing electron momenta respectively.

We begin with the fermion 1-particle irreducible 2-point function to the one loop level.



$$-i\Sigma(\not{p}) = g^2 \int \frac{d^dk}{(2\pi)^d} \frac{i}{((p-k)^2 - M^2 + i\epsilon)} \frac{\gamma^5 i(\not{k} + m)\gamma^5}{(k^2 - m^2 + i\epsilon)} + i\left(\not{p}\delta_\psi - \delta_m\right)$$

Since $\gamma^5\gamma^5 = 1$ and $\gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$, introducing a Feynman parameterization, we have:

$$\Sigma(\not{p}) = ig^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{(\not{k} - m)}{\left(x((p-k)^2 - M^2) + (1-x)(k^2 - m^2) + i\epsilon\right)^2} - (\not{p}\delta_\psi - \delta_m)$$

Defining $l = k - xp$ and $\Delta = -x(1-x)p^2 + xM^2 + (1-x)m^2$,

$$\Sigma(\not{p}) = ig^2 \int_0^1 dx \int \frac{d^dl}{(2\pi)^d} \frac{(\not{l} + x\not{p} - m)}{\left(l^2 - \Delta + i\epsilon\right)^2} - (\not{p}\delta_\psi - \delta_m)$$

Wick rotating and ignoring terms linear in l :

$$\Sigma(\not{p}) = -g^2 \int_0^1 dx \int \frac{d^dl_E}{(2\pi)^d} \frac{(x\not{p} - m)}{\left(l_E^2 + \Delta\right)^2} - (\not{p}\delta_\psi - \delta_m)$$

Integrating,

$$\Sigma(\not{p}) = -g^2 \int_0^1 dx \frac{\Gamma(2 - d/2)}{(4\pi)^2} \left(\frac{4\pi}{\Delta}\right)^{(2-d/2)} (x\not{p} - m) - (\not{p}\delta_\psi - \delta_m)$$

Thus,

$$\Sigma(m) = -g^2 \int_0^1 dx \frac{\Gamma(2 - d/2)}{(4\pi)^2} \left(\frac{4\pi}{(1-x)^2 m^2}\right)^{(2-d/2)} (xm - m) - (m\delta_\psi - \delta_m)$$

Considering only the divergent contributions, we substitute $\epsilon = (4 - d)$, and $\Gamma(2 - d/2) = \Gamma(\epsilon/2) = 2/\epsilon - \gamma + \mathcal{O}(\epsilon)$:

$$\Sigma(m) \sim \frac{mg^2}{(4\pi)^2} \frac{2}{\epsilon} \int_0^1 dx\, (1 - x) - (m\delta_\psi - \delta_m) = 0$$

Thus,

$$(m\delta_\psi - \delta_m) \sim \frac{mg^2}{(4\pi)^2} \frac{1}{\epsilon}$$

Now, after some algebra,

$$\frac{d}{d\not{p}}\Sigma(\not{p})\Big|_{\not{p}=m} = -g^2 \frac{\Gamma(2 - d/2)}{(4\pi)^2} \int_0^1 dx \left(\frac{4\pi}{(1-x)^2 m^2}\right)^{(2-d/2)} (x + (2 - d/2)2x) - \delta_\psi$$

Again considering only the divergent contributions,

$$\frac{d}{d\not{p}}\Sigma(\not{p})\big|_{\not{p}=m} \;\sim\; \frac{-g^2}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dx\, x - \delta_\psi = 0$$

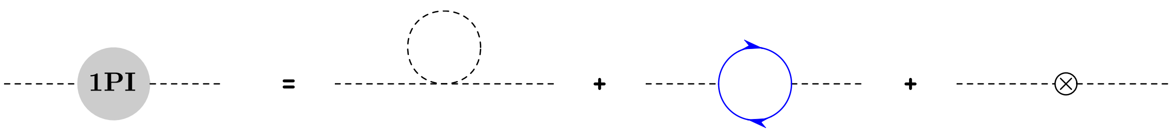
Thus,

$$\delta_\psi \;\sim\; -\frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}$$

And therefore,

$$\delta_m \;\sim\; -2m\,\frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}$$

We now consider the 1-particle irreducible boson 2-point function to the one loop level:



$$-i\mathcal{M}^2(p^2) = \frac{-i\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-M^2+i\epsilon} + i\left(p^2\delta_\phi - \delta_M\right)$$
$$+g^2\int\frac{d^dk}{(2\pi)^d}\frac{tr\left((\not{k}+\not{p}+m)\gamma^5(\not{k}+m)\gamma^5\right)}{((k+p)^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}$$

Now,

$$\frac{-i\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-M^2+i\epsilon} = \frac{-i\lambda}{2}\int\frac{d^dl_E}{(2\pi)^d}\frac{1}{l_E^2+M^2} = \frac{-i\lambda}{2(4\pi)}\Gamma(1-d/2)\left(\frac{4\pi}{M^2}\right)^{(1-d/2)}$$

From $\Gamma(\epsilon/2-1) = -2/\epsilon + \gamma - 1 + \mathcal{O}(\epsilon)$:

$$\frac{-i\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-M^2+i\epsilon} \sim \frac{i\lambda}{(4\pi)^2}\left(\frac{M^2}{\epsilon}\right)$$

Also,

$$g^2\int\frac{d^dk}{(2\pi)^d}\frac{tr\left((\not{k}+\not{p}+m)\gamma^5(\not{k}+m)\gamma^5\right)}{((k+p)^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}$$
$$= -g^2\int_0^1 dx\,\int\frac{d^dk}{(2\pi)^d}\frac{tr\left((\not{k}+\not{p}+m)(\not{k}-m)\right)}{\left(x((k+p)^2-m^2)+(1-x)(k^2-m^2)+i\epsilon\right)^2}$$

Now,

$$tr\left((\not{k}+\not{p}+m)(\not{k}-m)\right) = 4\left(k\cdot(p+k)-m^2\right)$$

Defining $l = k + xp$, ignoring terms linear in l , Wick rotating and defining $\Delta = m^2 - x(1-x)p^2$ this expression becomes :

$$= -4g^2\int_0^1 dx\,\int\frac{d^dl}{(2\pi)^d}\frac{\left((l-xp)\cdot(l+(1-x)p)-m^2\right)}{\left(l^2+x(1-x)p^2-m^2+i\epsilon\right)^2}$$

$$= 4ig^2\int_0^1 dx\,\int\frac{d^dl_E}{(2\pi)^d}\frac{\left(l_E^2+x(1-x)p^2+m^2\right)}{\left(l_E^2+\Delta\right)^2}$$

$$= \frac{4ig^2}{(4\pi)^2}\Gamma(1-d/2)\int_0^1 dx\ \left(\frac{4\pi}{\Delta}\right)^{(2-d/2)}\left(\frac{d}{2}\Delta + (1-d/2)(x(1-x)p^2+m^2)\right)$$

Concentrating on the divergent part this becomes:

$$\sim \frac{-4ig^2}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dx\ \left(2\Delta - (x(1-x)p^2+m^2)\right)$$

$$= \frac{-4ig^2}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dx\ \left(m^2 - 3x(1-x)p^2\right) = \frac{-2ig^2}{(4\pi)^2}\frac{2}{\epsilon}\left(2m^2-p^2\right)$$

Thus in terms of the divergent part:

$$-i\mathcal{M}^2(p^2) \sim \left(\frac{i}{(4\pi)^2\epsilon}\right)\left(\lambda M^2 - 4g^2\left(2m^2-p^2\right)\right) + i\left(p^2\delta_\phi - \delta_M\right)$$

We now apply the renormalization conditions:

$$-i\mathcal{M}^2(M^2) \sim \left(\frac{i}{(4\pi)^2\epsilon}\right)\left(\lambda M^2 - 4g^2\left(2m^2-M^2\right)\right) + i\left(M^2\delta_\phi - \delta_M\right) = 0$$

Thus,

$$\left(M^2\delta_\phi - \delta_M\right) = \left(\frac{1}{(4\pi)^2\epsilon}\right)\left(4g^2\left(2m^2-M^2\right) - \lambda M^2\right)$$

Also,

$$\frac{d}{dp^2}\mathcal{M}^2(p^2)\,|_{p^2=M^2}\ \sim\ \frac{4ig^2}{(4\pi)^2\epsilon} + i\delta_\phi = 0$$

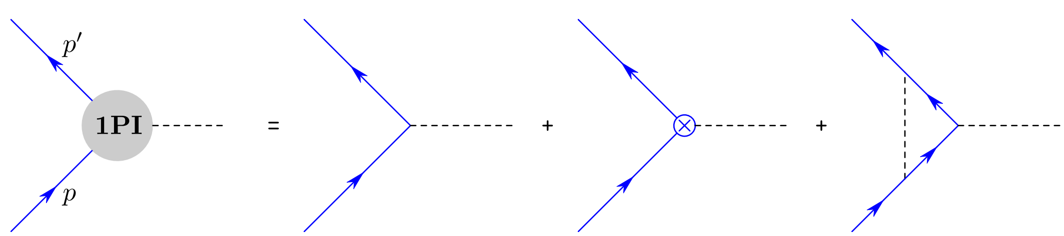
Thus,

$$\delta_\phi = \frac{-4g^2}{(4\pi)^2}\frac{1}{\epsilon}$$

and,

$$\delta_M = \left(\frac{1}{(4\pi)^2\epsilon}\right)\left(\lambda\, M^2 - 8\, g^2 m^2\right)$$

We now consider the 2-fermion 1-pseudoscalar vertex to the one loop level:



Which is given by:

$$g\Gamma = g\gamma^5 + g\delta\Gamma + g\delta_1\gamma^5$$

We may choose to set $p = p'$ since the renormalization condition is:

$$g\Gamma = g\gamma^5 \quad \text{at} \quad p' - p = q = 0$$

Which leads to:

$$\gamma^5\delta_1 = -\delta\Gamma(p,p)$$

$\overline{u}(p)\delta\Gamma(p,p)u(p)$

$$= g^2 \int \frac{d^dk}{(2\pi)^d} \frac{i}{((k-p)^2 - M^2 + i\epsilon)} \overline{u}(p)\gamma^5 \frac{i(\not{k}' + m)}{(k'^2 - m^2 + i\epsilon)} \gamma^5 \frac{i(\not{k} + m)}{(k^2 - m^2 + i\epsilon)} \gamma^5 u(p)$$

$$= -ig^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{((k-p)^2 - M^2 + i\epsilon)} \overline{u}(p) \frac{(\not{k} - m)}{(k^2 - m^2 + i\epsilon)} \gamma^5 \frac{(\not{k} - m)}{(k^2 - m^2 + i\epsilon)} u(p)$$

Where $k' = k + q$ and $q = p' - p$. Implementing the Feynman parameter trick as above:

$$\overline{u}(p)\delta\Gamma(p,p)u(p) = -i2g^2 \int_0^1 dz\,(1-z) \int \frac{d^dk}{(2\pi)^d} \frac{\overline{u}(p)(\not{k} - m)\gamma^5(\not{k} - m)u(p)}{(k^2 - 2zk\cdot p + (2z-1)m^2 - zM^2 + i\epsilon)^3}$$

Defining $l = k - zp$ and $\Delta = (1-z)^2m^2 + zM^2$ we have:

$$\overline{u}(p)\delta\Gamma(p,p)u(p) = -i2g^2 \int_0^1 dz\,(1-z) \int \frac{d^dl}{(2\pi)^d} \frac{N}{(l^2 - \Delta + i\epsilon)^3}$$

Where,

$$N = \overline{u}(p)(\not{l} + z\not{p} - m)\gamma^5(\not{l} + z\not{p} - m)u(p)$$

$$N = \overline{u}(p)(\not{l} - m(1-z))\gamma^5(\not{l} - m(1-z))u(p)$$

Ignoring terms linear in l , we have:

$$N = \overline{u}(p)\left(\not{l}\gamma^5\not{l} + m^2(1-z)^2\gamma^5\right)u(p)$$

We also make the replacement:

$$\not{l}\gamma^5\not{l} \to \frac{-1}{d} l^2 \eta_{\alpha\beta}\,\gamma^5\gamma^\alpha\gamma^\beta = -l^2\gamma^5$$

$$\overline{u}(p)\delta\Gamma(p,p)u(p) = i2g^2\overline{u}(p)\gamma^5 u(p) \int_0^1 dz\,(1-z) \int \frac{d^dl}{(2\pi)^d} \frac{l^2 - m^2(1-z)^2}{(l^2 - \Delta + i\epsilon)^3}$$

Thus,

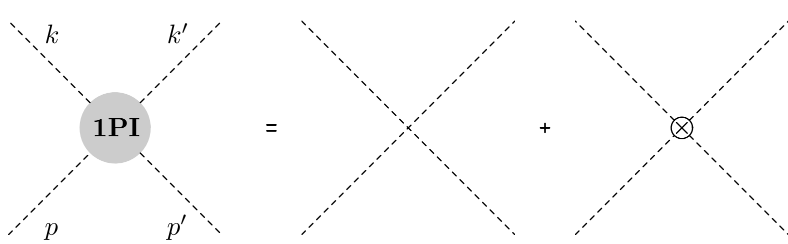
$$\delta_1 = 2g^2 \int_0^1 dz\,(1-z) \int \frac{d^dl_E}{(2\pi)^d} \frac{l_E^2 + m^2(1-z)^2}{(l_E^2 + \Delta)^3}$$

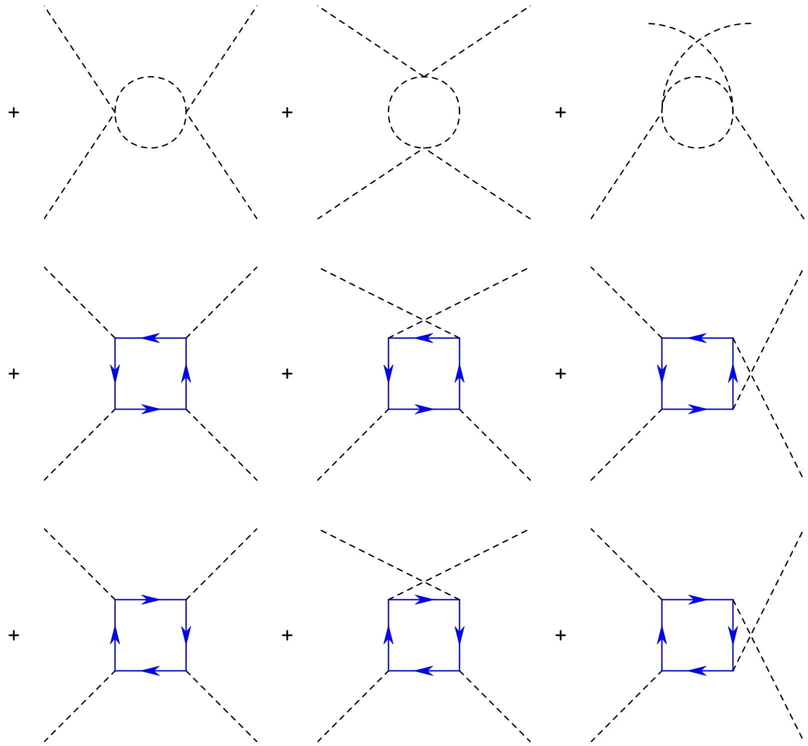
$$\delta_1 = \frac{g^2}{(4\pi)^2}\Gamma(2 - d/2) \int_0^1 dz\,(1-z) \left(\frac{4\pi}{\Delta}\right)^{(2-d/2)} \left(\frac{d}{2} + \frac{(2-d/2)m^2(1-z)^2}{(1-z)^2m^2 + zM^2}\right)$$

Taking the divergent part:

$$\delta_1 \sim \frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}$$

We now consider the 4-pseudoscalar vertex to the one loop level:





Which we will denote by:

$$-i\mathcal{A} = -i\lambda + (-i\lambda)^2\left(iV(s) + iV(t) + iV(u)\right) + i\mathcal{V}(p,p',k,k') - i\delta_\lambda$$

The normalization condition, in terms of the Mandelstam variables (s,t,u), is as follows:

$$-i\mathcal{A} = -i\lambda \quad \text{at} \quad s^2 = 4M^2\ ,\ t = u = 0$$

Note that $s = (p+p')^2 = (k+k')^2$, $t = (k-p)^2 = (k'-p')^2$ and $u = (k'-p)^2 = (k-p')^2$. When convenient we will explicitly work in the frame:

$$p_0 = p = p' = k = k' = (M,0)$$

Thus, denoting $\mathcal{V}(p_0,p_0,p_0,p_0) = \mathcal{V}(M)$, we have:

$$\delta_\lambda = -\lambda^2\left(V(4M^2) + 2V(0)\right) + \mathcal{V}(M)$$

Where,

$$iV(p^2) = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{i}{(k^2 - M^2 + i\epsilon)}\frac{i}{((k+p)^2 - M^2 + i\epsilon)}$$

$$V(p^2) = \frac{i}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2 + 2xk\cdot p + xp^2 - M^2 + i\epsilon\right)^2}$$

Defining $l = k + xp$ and $\Delta = -x(1-x)p^2 + M^2$:

$$V(p^2) = \frac{-1}{2}\int_0^1 dx \int \frac{d^dl_E}{(2\pi)^d}\frac{1}{\left(l_E^2 + \Delta\right)^2} = \frac{-1}{2(4\pi)^2}\Gamma(2 - d/2)\int_0^1 dx\ \left(\frac{4\pi}{\Delta}\right)^{(2-d/2)}$$

Extracting the divergent part (which is independent of p^2):

$$V(p^2) \sim \frac{-1}{\epsilon(4\pi)^2}$$

$$\delta_\lambda \sim \frac{3\lambda^2}{\epsilon(4\pi)^2} + \mathcal{V}(M)$$

We now compute $\mathcal{V}(M)$ (ignoring $i\epsilon$):

$$i\mathcal{V}(M) \ = \ -6\, g^4 \int \frac{d^dq}{(2\pi)^d} \frac{N}{D}$$

Where,

$$D = ((q-p')^2 - m^2)((q+p-k)^2 - m^2)((q+p)^2 - m^2)(q^2 - m^2)$$

and,

$$N = tr\left[\gamma^5(\not{q} - \not{p}' + m)\gamma^5(\not{q} + \not{p} - \not{k} + m)\gamma^5(\not{q} + \not{p} + m)\gamma^5(\not{q} + m)\right]$$

We first attempt to simplify D:

$$D = ((q-p')^2 - m^2)((q+p-k)^2 - m^2)((q+p)^2 - m^2)(q^2 - m^2)$$

Introducing Feynman parameters,

$$\frac{1}{D} = \int_0^1 dxdydzdw\, \delta(x+y+z+w-1)\, \frac{3!}{\widetilde{D}^4}$$

Where,

$$\widetilde{D} = x((q-p')^2 - m^2) + y((q+p-k)^2 - m^2) + z((q+p)^2 - m^2) + w(q^2 - m^2)$$

Replacing $p_0 = p = p' = k = k'$

$$\widetilde{D} = q^2 + M^2(z+x) + (z-x)(2q{\cdot}p_0) - m^2$$

Now,

$$\int_0^1 dxdydzdw\, \delta(x+y+z+w-1)\, f(x,z) \ = \ \int_0^1 dx \int_0^{1-x} dz\, (1-z-x)\, f(x,z)$$

Thus,

$$\frac{1}{D} = \int_0^1 dx \int_0^{1-x} dz\, \frac{6\,(1-z-x)}{\left(q^2 + M^2(z+x) + (z-x)(2q{\cdot}p_0) - m^2\right)^4}$$

We now consider N:

$$N = tr\left[(\not{q} - \not{p}_0 - m)(\not{q} + m)(\not{q} + \not{p}_0 - m)(\not{q} + m)\right]$$

We define $\alpha = q - p_0$ and $\beta = q + p_0$. Since the trace of an odd number of γ matrices vanishes:

$$N = tr\left[(\not{\alpha} - m)(\not{q} + m)(\not{\beta} - m)(\not{q} + m)\right] = N_4 + N_2 + N_0$$

$$N_4 + N_0 = tr\left[\not{\alpha}\not{q}\not{\beta}\not{q}) + m^4\right]$$

$$N_2 = m^2 tr\left[\not{\alpha}\not{\beta} + \not{q}\not{q} + (\not{\beta} - \not{\alpha})\not{q} - (\not{\alpha} + \not{\beta})\not{q}\right]$$

Or,

$$N_4 + N_0 = 4\left(2\alpha{\cdot}q\beta{\cdot}q - \alpha{\cdot}\beta q^2 + m^4\right)$$

$$N_2 = 4m^2\left(\alpha{\cdot}\beta - q^2 + 2p_0{\cdot}q\right)$$

Now, $\alpha{\cdot}\beta = q^2 - M^2$ and $\alpha{\cdot}q\,\beta{\cdot}q = (q^2)^2 - (p_0{\cdot}q)^2$.

Thus,

$$N = 4\left(q^2\left(q^2 + M^2\right) - 2(p_0{\cdot}q)^2 + m^4\right) + 4m^2\left(2p_0{\cdot}q - M^2\right)$$

Defining $l = q + (z-x)p_0$ and $\Delta = m^2 - (z+x)M^2 + (z-x)^2M^2$:

$$i\mathcal{V}(M) \ = \ -6\, g^4 \int \frac{d^dl}{(2\pi)^d} \int_0^1 dx \int_0^{1-x} dz\, \frac{6\,(1-z-x)\,N}{\left(\, l^2 - \Delta + i\epsilon\,\right)^4}$$

To compute the divergent part we take only the $(l^2)^2$ term in N.

$$i\mathcal{V}(M) \,\sim\, -(24)\,6\,g^4 \int \frac{d^dl}{(2\pi)^d} \int_0^1 dx \int_0^{1-x} dz\,(1-z-x)\,\frac{(l^2)^2}{(l^2-\Delta+i\epsilon)^4}$$

Or,

$$i\mathcal{V}(M) \,\sim\, -(24)\,6\,g^4 \int_0^1 dx \int_0^{1-x} dz\,(1-z-x)\,\frac{i}{(4\pi)^2}\,\Gamma(2-d/2)\left(\frac{4\pi}{\Delta}\right)^{(2-d/2)}$$

Thus,

$$\mathcal{V}(M) \,\sim\, \frac{-24\,g^4}{(4\pi)^2}\,\frac{2}{\epsilon}$$

Finally,

$$\delta_\lambda \sim \frac{\left(\,3\lambda^2 - 48\,g^4\,\right)}{2(4\pi)^2}\,\frac{2}{\epsilon}$$