QFT 2 : Problem Set 4

1 Peskin & Schroeder 11.3 : The Gross-Neveu model

We have the following Lagrangian in $d = 2$ dimensions :

$$\mathcal{L} \;=\; \overline{\psi}_i \, i\partial\!\!\!/ \, \psi_i \;+\; \frac{1}{2}\, g^2 \left(\overline{\psi}_i \psi_i\right)^2$$

with $i = 1, \ldots, N$. The two-dimensional Dirac algebra is represented by γ^μ :

$$\{\gamma^\mu, \gamma^\nu\} \;=\; 2\,\eta^{\mu\nu} 1$$

$$\gamma^0 = \sigma^2 \qquad \text{and} \qquad \gamma^1 = i\sigma^1$$

We also define $\gamma^5 = \gamma^0\gamma^1 = \sigma^3$.

(a)

We consider the behavior of a massive generalization of this Lagrangian ,

$$\mathcal{L} \;=\; \overline{\psi}_i \, i\partial\!\!\!/ \, \psi_i \;+ m\overline{\psi}_i \psi_i \;+\; \frac{1}{2}\, g^2 \left(\overline{\psi}_i \psi_i\right)^2$$

under the transformation :

$$\psi_i \;\rightarrow\; \gamma^5 \, \psi_i$$

We have :

$$\mathcal{L} \;\rightarrow\; \left(\gamma^5 \psi_i\right)^\dagger \gamma^0 \, i\partial\!\!\!/ \, \gamma^5 \psi_i \;+\; m \left(\gamma^5 \psi_i\right)^\dagger \gamma^0 \gamma^5 \psi_i \;+\; \frac{1}{2}\, g^2 \left(\left(\gamma^5 \psi_i\right)^\dagger \gamma^0 \gamma^5 \psi_i \right)^2$$

Since,

$$\left\{\gamma^\mu, \gamma^5\right\} \;=\; 0 \qquad \text{and} \qquad \gamma^5\gamma^5 \;=\; 1 \qquad \text{and} \qquad \left(\gamma^5\right)^\dagger \;=\; \gamma^5$$

We find :

$$\mathcal{L} \;\rightarrow\; \psi_i^\dagger \gamma^5 \gamma^0 \gamma^\mu \, i\partial_\mu \gamma^5 \psi_i \;+ m\psi_i^\dagger \gamma^5 \gamma^0 \gamma^5 \psi_i \;+\; \frac{1}{2}\, g^2 \left(\psi_i^\dagger \gamma^5 \gamma^0 \gamma^5 \psi_i \right)^2$$

Or,

$$\mathcal{L} \;\rightarrow\; \overline{\psi}_i \, i\partial\!\!\!/ \, \psi_i \;-\; m\overline{\psi}_i \psi_i \;+\; \frac{1}{2}\, g^2 \left(\overline{\psi}_i \psi_i\right)^2$$

Thus only the mass term violates the symmetry.

(b)

Denoting the mass dimension of the Lagrangian by $[\,\mathcal{L}\,]$ we have :

$$[\,\mathcal{L}\,] \;=\; d$$

Thus,

$$[\,\psi_i\,] \;= (d-1)\,/2 \qquad \text{and} \qquad 2\,[\,g\,] \;= d - 2\,(d-1)$$

Or,

$$[\,g\,] \;=\; 1 - d/2$$

Thus in $d = 2$ dimensions the coupling is dimensionless and therefore the theory is renormalizable.

(c)

We consider the following path integral :

$$\int \mathcal{D}\overline{\psi}\,\mathcal{D}\psi\,\mathcal{D}\sigma\;\exp\left[\,i\int d^2x\,\left\{\overline{\psi}_i\,i\partial\!\!\!/\,\psi_i\;-\;\sigma\,\overline{\psi}_i\psi_i\;-\;\frac{1}{2}\,g^{-2}\,\sigma^2\right\}\right]$$

Introducing the change of variables :

$$\widetilde{\sigma}\;=\;\sigma\;+\;g^2\,\overline{\psi}_i\psi_i$$

We find :

$$\int \mathcal{D}\overline{\psi}\,\mathcal{D}\psi\,\mathcal{D}\widetilde{\sigma}\;\exp\left[\,i\int d^2x\,\left\{\overline{\psi}_i\,i\partial\!\!\!/\,\psi_i\;+\;\frac{1}{2}\,g^2\,\left(\overline{\psi}_i\psi_i\right)^2\;-\;\frac{1}{2}\,g^{-2}\,\widetilde{\sigma}^2\right\}\right]$$

$$=\int \mathcal{D}\overline{\psi}\,\mathcal{D}\psi\;e^{\,i\int d^2x\,\mathcal{L}}\int \mathcal{D}\widetilde{\sigma}\;\exp\left[\,i\int d^2x\,\left\{-\,\frac{1}{2}\,g^{-2}\,\widetilde{\sigma}^2\right\}\right]$$

Thus up to a constant given by,

$$\left[\det\left(g^{-2}\right)\right]^{-1/2}\;=\;\int \mathcal{D}\sigma\;\exp\left[\,i\int d^2x\,\left\{-\,\frac{1}{2}\,g^{-2}\,\sigma^2\right\}\right]$$

the functional integrals are identical.

(d)

We define the generating functional for the scalar field $\sigma(x)$:

$$Z\,[J]\;=\;\int \mathcal{D}\,\overline{\psi}\mathcal{D}\,\psi\mathcal{D}\sigma\;e^{iS\left[\sigma,\overline{\psi},\psi\right]+i\int d^2x\,J\,\sigma}$$

Where,

$$S\left[\sigma,\overline{\psi},\psi\right]\;=\;\int d^2x\,\left(\overline{\psi}_i\,i\partial\!\!\!/\,\psi_i\;-\;\sigma\,\overline{\psi}_i\psi_i\;-\;\frac{1}{2}\,g^{-2}\,\sigma^2\right)$$

We now decompose σ as $\sigma\;=\;\sigma_c+\eta$. This leads to :

$$Z\,[J]\;=\;e^{i\,\widetilde{S}[\sigma_c]\,+\,i\,\int d^2x\,J\,\sigma_c}\int \mathcal{D}\eta\;e^{i\,\widetilde{S}[\eta]\,-\,i\,g^{-2}\,\int d^2x\,\sigma_c\,\eta\,+\,i\,\int d^2x\,J\,\eta}$$

$$\times\int \mathcal{D}\overline{\psi}\,\mathcal{D}\psi\;e^{i\,\int\,d^2x\,\left(\overline{\psi}_i\,i\partial\!\!\!/\,\psi_i\,-\,\sigma_c\,\overline{\psi}_i\psi_i\,-\,\eta\,\overline{\psi}_i\psi_i\right)}$$

Where,

$$\widetilde{S}\,[\sigma]\;=\;\frac{-1}{2\,g^2}\int d^2x\,\sigma^2$$

Since we are computing the effective action to the one loop level we may ignore the $\eta\overline{\psi}\psi$ term in the Lagrangian. From the definition :

$$\left(\det\left(\,i\partial\!\!\!/\;-\;\sigma_c\right)\right)^N\;=\;\int \mathcal{D}\overline{\psi}\,\mathcal{D}\psi\;\exp\left[\,i\int d^2x\,\left\{\overline{\psi}_i\,i\partial\!\!\!/\,\psi_i\;-\;\sigma_c\,\overline{\psi}_i\psi_i\,\right\}\right]$$

We have :

$$Z\,[J]\;=\;e^{i\,\widetilde{S}[\sigma_c]\,+\,i\,\int d^2x\,J\,\sigma_c}\;\left(\det\left(\,i\partial\!\!\!/\;-\;\sigma_c\right)\right)^N$$

$$\times\int \mathcal{D}\eta\;e^{i\,\widetilde{S}[\eta]\,-\,i\,g^{-2}\,\int d^2x\,\sigma_c\,\eta\,+\,i\,\int d^2x\,J\,\eta}$$

We can also ignore terms linear in η since they will vanish by the condition we place on the external current counterterm that all 1PI one-point graphs vanish as well as the condition :

$$\frac{\delta S_1}{\delta \eta(x)} \;+\; J_1(x) \;=\; 0$$

Where J_1 is the renormalized external current and $S_1[\eta]$ is the renormalized action appearing in the η path integral. Furthermore since there will be no contribution to the effective action to one loop from the η path integral we may write :

$$\Gamma[\sigma_c] \;=\; -i\,\ln Z[J] \;-\; \int d^2x\, J(x)\sigma_c(x)$$

Or,

$$\Gamma[\sigma_c] \;=\; \widetilde{S}[\sigma_c] \;-i\,N\; \ln\det(\, i\partial\!\!\!/ \;-\; \sigma_c)$$

Where counterterms are implicit. We now calculate :

$$\ln\,\det\,(\, i\partial\!\!\!/ \;-\; \sigma_c) \;=\; \mathrm{Tr}\left[\ln(\, i\partial\!\!\!/ \;-\; \sigma_c)\right]$$

The determinant is invariant under a (unitary) Fourier transform. Thus, denoting

$$D(x,y) \;=\; (\, i\partial\!\!\!/_x \;-\; \sigma_c)\; \delta(x-y)$$

we have :

$$\ln\left[\det D[\sigma_c]\right] \;=\; \ln\left[\det \widehat{D}[\sigma_c]\right]$$

Where,

$$\widehat{D}(p,k) \;=\; \int d^2x\, e^{i\,p\,x}\;(\, i\partial\!\!\!/_x \;-\; \sigma_c)\; e^{-i\,k\,x} \;=\; (2\pi)^2\;\delta^2(p-k)\;(p\!\!\!/ \;-\; \sigma_c)$$

Thus,

$$\mathrm{Tr}\left[\ln\; D\right] \;=\; \mathrm{Tr}\left[\ln\; \widehat{D}\right] \;=\; \mathrm{Tr}\left[(2\pi)^2\;\delta^2(p-k)\;\ln\;(p\!\!\!/ \;-\; \sigma_c)\right]$$

Or,

$$\mathrm{Tr}\left[\ln\; D\right] \;=\; (2\pi)^2\;\delta^2(0)\int \frac{d^2p}{(2\pi)^2}\;\mathrm{tr}\left[\ln\;(p\!\!\!/ \;-\; \sigma_c)\right]$$

Where we use Tr for the full functional trace and tr for the trace over Dirac indices. We now find the eigenvalues of :

$$p\!\!\!/ \;-\; \sigma_c \;=\; \begin{pmatrix} -\sigma_c & -i\left(p^0 - p^1\right) \\ i\left(p^0 + p^1\right) & -\sigma_c \end{pmatrix}$$

It turns out that the eigenvalues are $-\sigma_c \pm \sqrt{p^2}$. Thus,

$$\mathrm{Tr}\left[\ln\; D\right] \;=\; (2\pi)^2\;\delta^2(0)\int \frac{d^2p}{(2\pi)^2}\;\mathrm{tr}\left[\begin{pmatrix} \ln\left(-\sigma_c + \sqrt{p^2}\right) & 0 \\ 0 & \ln\left(-\sigma_c - \sqrt{p^2}\right) \end{pmatrix}\right]$$

Or,

$$\mathrm{Tr}\left[\ln\; D\right] \;=\; V\,T\int \frac{d^2p}{(2\pi)^2}\;\ln\left(-p^2 + \sigma_c^2\right)$$

Now, introducing $\epsilon = 2 - d$,

$$\int \frac{d^dp}{(2\pi)^d}\;\ln\left(-p^2 + \sigma_c^2\right) \;=\; i\,\frac{2}{d}\,\Gamma\left[1 - d/2\right]\left(\frac{\sigma_c^2}{4\pi}\right)^{d/2} \;=\; i\,\left(1 + \frac{\epsilon}{2}\right)\,\Gamma\left[\epsilon/2\right]\left(\frac{\sigma_c^2}{4\pi}\right)^{1-\epsilon/2}$$

$$=\; i\,\left(1 + \frac{\epsilon}{2}\right)\left(\frac{2}{\epsilon} - \gamma\right)\left(\frac{\sigma_c^2}{4\pi}\right)\left(1 - \frac{\epsilon}{2}\ln\left(\frac{\sigma_c^2}{4\pi}\right)\right) \;=\; i\,\left(\frac{\sigma_c^2}{4\pi}\right)\left(1 - \ln\left(\sigma_c^2\right) + \left(\frac{2}{\epsilon} - \gamma + \ln\,4\pi\right)\right)$$

Applying modified minimal subtraction, we find :

$$\Gamma[\sigma_c] \;=\; -V\,T\,\left[g^{-2}\,\sigma_c^2/2 \,+\, N\left(\frac{\sigma_c^2}{4\pi}\right)\left(\ln\left(\sigma_c^2/M^2\right)-1\right)\right]$$

Where M^2 is the renormalization scale. Thus, to one loop,

$$V_{eff} \;=\; g^{-2}\,\sigma_c^2/2 \,+\, N\left(\frac{\sigma_c^2}{4\pi}\right)\left(\ln\left(\sigma_c^2/M^2\right)-1\right)$$

(e)

We minimise the effective potential to find :

$$0 = \frac{\partial V_{eff}}{\partial \sigma_c} \;=\; g^{-2}\,\sigma_c \,+\, N\left(\frac{\sigma_c}{2\pi}\right)\,\ln\left(\sigma_c^2/M^2\right)$$

This has solution :

$$\sigma_c^2 \;=\; M^2\,e^{-2\pi/(g^2N)}$$

This gives a mass to the fermion and breaks the symmetry in part (a). This breaking will take place for any non-zero value of M. Note that the mass goes to zero for small g. Note also that this mass is non-perturbative in the coupling g, that is it does not have a well defined expansion around $g = 0$, and thus could not be computed using an expansion in Feynman diagrams.

(f)

Clearly we may write :

$$V_{eff} \;=\; N\left(\frac{\sigma_c^2}{2g^2N} \,+\, \left(\frac{\sigma_c^2}{4\pi}\right)\left(\ln\left(\sigma_c^2/M^2\right)-1\right)\right)$$

Which is of the form :

$$V_{eff} \;=\; N\,f(\,g^2N\,)$$

In fact, returning to the original path integral after the introduction of η :

$$Z[J] \;=\; e^{i\,\widetilde{S}[\sigma_c]\,+\,i\,\int d^2x\,J\,\sigma_c}\int \mathcal{D}\eta\,e^{i\,\widetilde{S}[\eta]\,-\,i\,g^{-2}\,\int d^2x\,\sigma_c\,\eta\,+\,i\,\int d^2x\,J\,\eta}$$

$$\times\int \mathcal{D}\overline{\psi}\,\mathcal{D}\psi\,e^{i\,\int\,d^2x\,\left(\overline{\psi}_i\,i\partial\!\!\!/\,\psi_i\,-\,\sigma_c\,\overline{\psi}_i\psi_i\,-\,\eta\,\overline{\psi}_i\psi_i\right)}$$

Ignoring terms linear in η , we see that this may be written as :

$$Z[J] \;=\; e^{i\,\widetilde{S}[\sigma_c]\,+\,i\,\int d^2x\,J\,\sigma_c}\int \mathcal{D}\eta\,\left(\det(\,i\partial\!\!\!/\,-\,(\sigma_c+\eta))\right)^N\,e^{i\,\widetilde{S}[\eta]}$$

Using,

$$\det(\,i\partial\!\!\!/\,-\,(\sigma_c+\eta)) \;=\; \det(\,i\partial\!\!\!/\,-\,\sigma_c)\,\det\left(1-(\,i\partial\!\!\!/\,-\,\sigma_c)^{-1}\,\eta\right)$$

We have,

$$Z[J] \;=\; e^{i\,\widetilde{S}[\sigma_c]\,+\,i\,\int d^2x\,J\,\sigma_c}\,\left(\det(\,i\partial\!\!\!/\,-\,\sigma_c)\right)^N\int \mathcal{D}\eta\,e^{i\,\widetilde{S}_2[\eta]}$$

Where $\widetilde{S}_2$ is the action that generates the two loop and above contributions to the effective action :

$$\widetilde{S}_2[\eta] \;=\; N\left[\frac{-1}{2\,g^2N}\int d^2x\,\eta^2 \,-\, i\,\mathrm{Tr}\left[\ln\left(1-(\,i\partial\!\!\!/\,-\,\sigma_c)^{-1}\,\eta\right)\right]\right]$$

Clearly this is also of the form :

$$\widetilde{S}_2[\eta] \;=\; N\,f(\,g^2N\,)$$

Thus as $N\to\infty$ the contributions to the path integral from these terms are suppressed.