QFT 2 : Problem Set 5

P & S 12.1 : Beta functions in Yukawa theory

We have the Lagrangian for a pseudoscalar Yukawa theory :

$$\mathcal{L} = \frac{1}{2}\,\partial^\mu\phi\partial_\mu\phi + \overline{\psi}(i\partial\!\!\!/)\psi - ig\phi\overline{\psi}\gamma^5\psi - \frac{\lambda}{4!}\phi^4$$

In Peskin & Schroeder 10.2 we computed the following divergent parts of the field renormalization counterterms :

$$\delta_\phi \;\sim\; \frac{-2g^2}{(4\pi)^2}\,\frac{2}{\epsilon} \qquad \text{and} \qquad \delta_\psi \;\sim\; \frac{-g^2}{2(4\pi)^2}\,\frac{2}{\epsilon}$$

Or, in the language of Peskin & Schroeder :

$$A_\phi \;=\; \frac{-2g^2}{(4\pi)^2} \qquad \text{and} \qquad A_\psi \;=\; \frac{-g^2}{2(4\pi)^2}$$

Also from Peskin & Schroeder 10.2 we computed the following divergent parts of the coupling constant counterterms :

$$\delta_\lambda \;\sim\; \frac{\left(3\lambda^2 - 48\,g^4\right)}{2(4\pi)^2}\,\frac{2}{\epsilon} \qquad \text{and} \qquad \delta_g \;=\; g\,\delta_1 \;\sim\; \frac{g^3}{(4\pi)^2}\,\frac{2}{\epsilon}$$

Again in the language of Peskin & Schroeder :

$$B_\lambda \;=\; \frac{\left(48\,g^4 - 3\lambda^2\right)}{2(4\pi)^2} \qquad \text{and} \qquad B_g \;=\; \frac{-g^3}{(4\pi)^2}$$

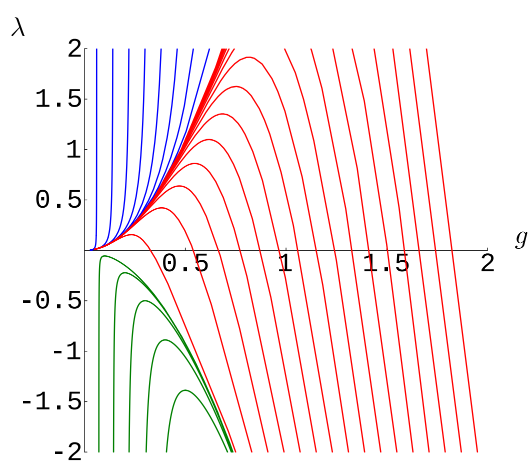
Thus,

$$\beta_\lambda \;=\; -2B_\lambda \,-\, 4\,\lambda\,A_\phi \;=\; \frac{1}{(4\pi)^2}\,\left(3\lambda^2 - 48\,g^4 + 8\,\lambda g^2\right)$$

And,

$$\beta_g \;=\; -2B_g - g\,(A_\phi + 2A_\psi) \;=\; \frac{1}{(4\pi)^2}\,\left(5\,g^3\right)$$

The following is a plot of the flow of the (β_g, β_λ) vector field. All curves flow to the ultraviolet in the direction of increasing g. Note the difference in the behavior of the blue, red and green curves.



P & S 12.2 : Beta function of the Gross-Neveu model

We begin with the Lagrangian from Peskin & Schroeder 11.3 that includes both scalars and fermions :

$$S\left[\sigma,\overline{\psi},\psi\right] \;=\; \int d^2x\,\left(\overline{\psi}_i\, i\partial\!\!\!/\,\psi_i \;-\; \sigma\,\overline{\psi}_i\psi_i \;-\; \frac{1}{2}\,g^{-2}\,\sigma^2\right)$$

Introducing the field redefinition $\sigma \;\rightarrow\; g\,\sigma$ we have the generating functional :

$$Z\left[J,\overline{\eta},\eta\right] \;=\; \int \mathcal{D}\,\overline{\psi}\mathcal{D}\,\psi\mathcal{D}\sigma\; e^{i\,\widetilde{S}\left[\sigma,\overline{\psi},\psi\right]+i\,\int d^2x\,\left(J\,\sigma\,+\,\overline{\eta}\,\psi\,+\,\overline{\psi}\,\eta\right)}$$

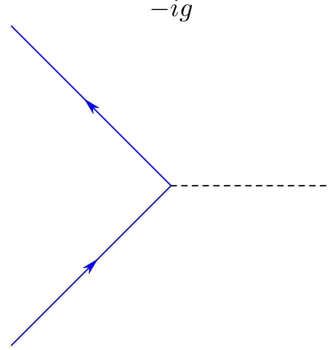
Where,

$$\widetilde{S}\left[\sigma,\overline{\psi},\psi\right] \;=\; \int d^2x\,\left(\overline{\psi}_i\, i\partial\!\!\!/\,\psi_i \;-\; \frac{1}{2}\,\sigma^2 \;-\; g\,\sigma\,\overline{\psi}_i\psi_i\right)$$

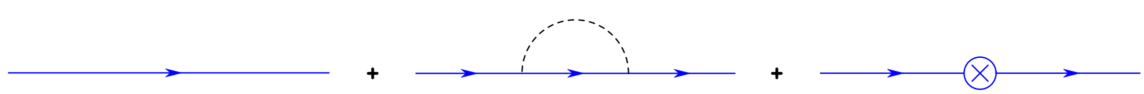
We have the following Feynman rules:

$$i\,(-1+i\epsilon)^{-1} \qquad\qquad i\,\delta_{ij}\;(p\!\!\!/ + i\epsilon)^{-1}$$

$$-ig$$



And similarly for the counterterm Feynmann rules. We consider the fermion propagator to one loop :

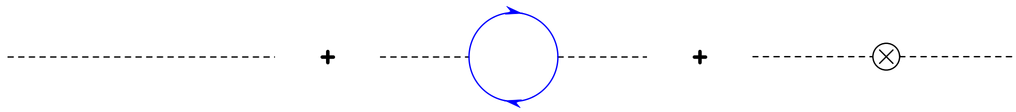


We compute the loop diagram :

$$-i\Sigma(p\!\!\!/) \;=\; (-ig)^2 \int \frac{d^dk}{(2\pi)^d}\,(-i)\,\frac{i(k\!\!\!/)}{(k^2+i\epsilon)} \;=\; 0$$

Thus $\delta_\psi = 0$.

We now consider the scalar propagator to one loop :



We compute the loop diagram :

$$-i\mathcal{M}^2(p^2) \;=\; -g^2\,N\int\frac{d^dk}{(2\pi)^d}\frac{tr\,(\not{k}\,(\not{k}+\not{p}))}{k^2\,(k+p)^2}$$

Introducing a Feynman parameter :

$$-i\mathcal{M}^2(p^2) \;=\; -g^2\,N\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{tr\,(\not{k}\,(\not{k}+\not{p}))}{(k^2+xp^2+2xk\cdot p)^2}$$

Substituting $l = k + xp$ and ignoring terms that do not contribute to the divergence :

$$-i\mathcal{M}^2(p^2) \;\sim\; -2\,g^2\,N\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{(l^2-\Delta)^2} \;=\; -2\,g^2\,N\int_0^1 dx\,\frac{-i}{4\pi}\,\frac{d}{2}\,\Gamma\left(1-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta}\right)^{1-d/2}$$

Where $\Delta = -x(1-x)p^2$. Thus the divergent part is ($\epsilon = 2-d$) :

$$-i\mathcal{M}^2(p^2) \;\sim\; \frac{i}{2\pi}\,g^2\,N\,\frac{2}{\epsilon}$$

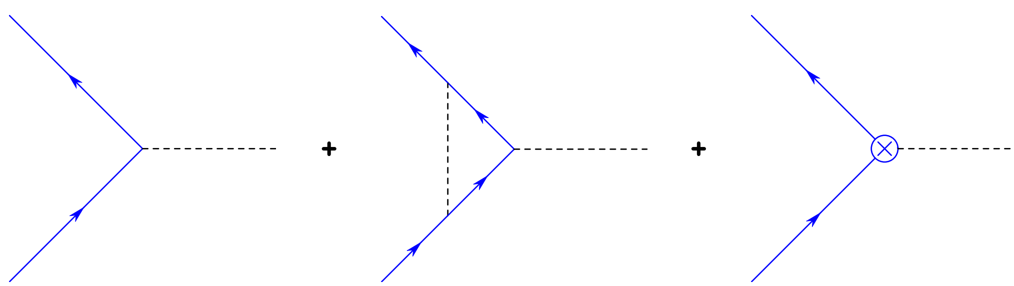
Or,

$$\delta_\sigma \;\sim\; N\,\frac{g^2}{2\pi}\,\ln\left(\frac{\Lambda^2}{M^2}\right)$$

Thus,

$$A_\sigma \;=\; N\,\frac{g^2}{2\pi} \qquad \text{and} \qquad A_\psi \;=\; 0$$

We now consider the scalar-fermion vertex:



We compute the loop diagram. We denote the incoming and out going fermion momenta as p and p' respectively. We denote the amputated greens function to this order by $-ig\Gamma$ where :

$$\Gamma = 1 + \delta\Gamma + \delta_1$$

We choose the renormalization condition :

$$-ig\Gamma = -ig \quad \text{at} \quad (p' - p)^2 = q^2 = -M^2$$

Which leads to:

$$\delta_1 = -\delta\Gamma(p+q,p)$$

Where

$$\overline{u}(p+q)\delta\Gamma(p+q,p)u(p) \;=\; -ig^2\int\frac{d^dk}{(2\pi)^d}\overline{u}(p+q)\frac{(\not k+\not q)}{(k+q)^2}\frac{\not k}{k^2}u(p)$$

Now, since the fermion field is massless,

$$\overline{u}(p+q)\not q \;=\; \overline{u}(p')\,(\not p' - \not p) \;=\; -\,\overline{u}(p+q)\not p$$

And,

$$-\not p\,\not k\,u(p) \;=\; -2\,p\cdot k\,u(p)$$

Thus,

$$\overline{u}(p+q)\delta\Gamma(p+q,p)u(p) \;=\; -ig^2\int\frac{d^dk}{(2\pi)^d}\overline{u}(p+q)\frac{(\not k\,\not k - 2\,p\cdot k\,)}{k^2\;(k+q)^2}u(p)$$

Again we are interested only in the divergent part of this expression. Thus we have :

$$\delta\Gamma(p+q,p) \;\sim\; -ig^2\int\frac{d^dk}{(2\pi)^d}\frac{\not k\,\not k}{k^2\;(k+q)^2} \;=\; -ig^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k+q)^2}$$

Or, choosing $l = k+q$:

$$\delta_1 \;\sim\; ig^2\int\frac{d^dl}{(2\pi)^d}\,\frac{1}{l^2} \;\sim\; \frac{g^2}{4\pi}\,\frac{2}{\epsilon}$$

Thus ($\delta_g = g\delta_1$):

$$B_g \;=\; -\,\frac{g^3}{4\pi}$$

Thus we have :

$$\beta_g \;=\; -2B_g - g\,(A_\sigma + 2A_\psi) \;=\; -\,\frac{g^3}{2\pi}\,(N-1)$$

Thus this theory is asymptotically free. A similar result may be derived from applying the Callan-Symanzik equation to the effective potential derived in P&S 11.3.

$$V_{eff} \;=\; g^{-2}\,\sigma_c^2/2 \;+\; N\,\left(\frac{\sigma_c^2}{4\pi}\right)\left(\ln\left(\sigma_c^2/M^2\right) - 1\right)$$

We solve

$$\left(M\,\frac{\partial}{\partial M} \;+\; \beta_g\,\frac{\partial}{\partial g}\right)V_{eff} \;=\; 0$$

Here we assume that the external sources are set to zero so that the effective potential is minimized (see P&S 13.25). This leads to

$$\beta_g \;=\; -\,N\,\frac{g^3}{2\pi}$$

Note that this result reflects the fact that the one-loop effective potential only involves N in the combination g^2N. To find the terms of $O(N^0)$ we would need to compute the two loop effective potential.