QFT 2 : Problem Set 6

1.) P & S 13.3 : The CP$^{\mathrm{N}}$ model

(a)

We begin with the Lagrangian

$$\mathcal{L}\,[z] \;=\; \frac{1}{g^2}\;[\,\partial\bar{z}_j\!\cdot\!\partial z_j \;+\; \bar{z}_j\,\bar{z}_k\,\partial z_j\!\cdot\!\partial z_k\,]$$

Given the constraint $\bar{z}_j z_j = 1$, this Lagrangian is invariant under

$$z_j(x) \;\rightarrow\; e^{i\alpha(x)}\,z_j(x)$$

This is demonstrated as follows

$$\begin{aligned} g^2\,\mathcal{L}\left[e^{i\alpha}z\right] &= (\partial\bar{z}_j - i\bar{z}_j\partial\alpha)\!\cdot\!(\partial z_j + iz_j\partial\alpha) \;+\; \bar{z}_j\,\bar{z}_k\;(\partial z_j + iz_j\partial\alpha)\!\cdot\!(\partial z_k + iz_k\partial\alpha) \\ &= g^2\,\mathcal{L}\,[z] \;+\; \bar{z}_j z_j\,(1-\bar{z}_k z_k)\;\partial\alpha\!\cdot\!\partial\alpha \;+\; (1-\bar{z}_k z_k)\;i\partial\alpha\!\cdot\!(z_j\partial\bar{z}_j - \bar{z}_j\partial z_j) \end{aligned}$$

Where we have applied the constraint $\bar{z}_j z_j = 1$, a consequence of which is $\bar{z}_j\partial z_j = -z_j\partial\bar{z}_j$. Thus the Lagrangian is invariant under the above local transformation. We now consider the non-linear sigma model action for $N = 3$:

$$\mathcal{L}_{S^2}\,[n] \;=\; \frac{1}{2g^2}\;\partial n^j\!\cdot\!\partial n^j \qquad \text{where} \qquad \sum_{j=1}^{N=3} n^j\,n^j \;=\; 1$$

Here we have denoted the target space of the non-linear sigma model to be S^2. We now introduce the target space coordinates $n^j = \bar{z}\sigma^i z$, where σ^i are the Pauli matrices and $z \in \mathbb{C}^2$. Making use of the Fierz identity

$$\sigma^i_{ab}\,\sigma^i_{cd} \;=\; \delta_{ab}\,\delta_{cd} \;-\; 2\,\epsilon_{ac}\,\epsilon_{bd}$$

We find

$$n^j\,n^j \;=\; \bar{z}_a z_b\,\bar{z}_c z_d\,\sigma^i_{ab}\,\sigma^i_{cd} \;=\; \bar{z}_a z_a\,\bar{z}_b z_b$$

Hence $n^j n^j = 1$ implies $\bar{z}_a z_a = 1$. We now make use of the Fierz identity to reexpress the nonlinear sigma model Lagrangian

$$g^2\,\mathcal{L}_{S^2} \;=\; \frac{1}{2}\,\partial(\bar{z}_a z_b)\!\cdot\!\partial(\bar{z}_c z_d)\,\sigma^i_{ab}\,\sigma^i_{cd} \;=\; -\,\partial(\bar{z}_a z_b)\!\cdot\!\partial(\bar{z}_c z_d)\;\epsilon_{ac}\,\epsilon_{bd}$$

Expressing the target space coordinates as $z_a = (a, b)$ we find

$$\begin{aligned} g^2\,\mathcal{L}_{S^2} &= 2\left[\,\partial(\bar{a}b)\!\cdot\!\partial\big(\bar{b}a\big) - \partial(\bar{a}a)\!\cdot\!\partial\big(\bar{b}b\big)\,\right] \\ &= 2\left[\,\bar{b}b\,\partial\bar{a}\!\cdot\!\partial a \;+\; \bar{a}a\,\partial\bar{b}\!\cdot\!\partial b \;-\; \bar{b}a\,\partial\bar{a}\!\cdot\!\partial b \;-\; \bar{a}b\,\partial\bar{b}\!\cdot\!\partial a\,\right] \end{aligned}$$

We now consider the Lagrangian for the CP^1 model. Making use of the constraint we write

$$\begin{aligned} g^2\,\mathcal{L}_{CP^1} &= \partial\bar{z}_j\!\cdot\!\partial z_j \;-\; z_j\,\bar{z}_k\,\partial\bar{z}_j\!\cdot\!\partial z_k \\ &= \bar{b}b\,\partial\bar{a}\!\cdot\!\partial a \;+\; \bar{a}a\,\partial\bar{b}\!\cdot\!\partial b \;-\; \bar{b}a\,\partial\bar{a}\!\cdot\!\partial b \;-\; \bar{a}b\,\partial\bar{b}\!\cdot\!\partial a \end{aligned}$$

Thus the models are equivalent. It should come as no surprise that the manifolds are the same, $S^2 = CP^1$.

(b)

We now consider the Lagrangian

$$g^2\,\mathcal{L}\left[z,A,\lambda\right] \,=\, \bar{D}\bar{z}_j\!\cdot\! Dz_j \,-\, \lambda\left(\bar{z}_j z_j - 1\right)$$

Where $D_\mu = \partial_\mu + iA_\mu$. We write out the generating functional and integrate over the scalar field λ to produce a functional delta function

$$\begin{aligned} Z \;&=\; \int \mathcal{D}\bar{z}\mathcal{D}z \int \mathcal{D}A\mathcal{D}\lambda\; \exp\left[\frac{i}{g^2}\int d^2x\,\bar{D}\bar{z}_j\!\cdot\! Dz_j \,-\, \lambda\left(\bar{z}_j z_j - 1\right)\right] \\ &=\; \int \mathcal{D}\bar{z}\mathcal{D}z \int \mathcal{D}A\,\delta\left[\bar{z}_j z_j - 1\right]\,\exp\left[\frac{i}{g^2}\int d^2x\,\bar{D}\bar{z}_j\!\cdot\! Dz_j\right] \end{aligned}$$

Using the constraint we may write

$$\begin{aligned} \bar{D}\bar{z}_j\!\cdot\! Dz_j \;&=\; \left(\partial\bar{z}_j - iA\,\bar{z}_j\right)\!\cdot\!\left(\partial z_j + iA\,z_j\right) \\ &=\; \partial\bar{z}_j\!\cdot\!\partial z_j \,+\, A^2 \,-\, 2i\,\bar{z}_j A\!\cdot\!\partial z_j \\ &=\; \partial\bar{z}_j\!\cdot\!\partial z_j \,+\, \left(A - i\bar{z}_j\partial z_j\right)^2 \,+\, \left(\bar{z}_j\partial z_j\right)^2 \end{aligned}$$

Changing variables in the A path integral and dropping the large constant term we find the generating functional of the CP^N model

$$Z \,=\, \int \mathcal{D}\bar{z}\mathcal{D}z\,\delta\left[\bar{z}_j z_j - 1\right]\,\exp\left[\frac{i}{g^2}\int d^2x\,\left[\,\partial\bar{z}_j\!\cdot\!\partial z_j \,+\, \bar{z}_j\bar{z}_k\,\partial z_j\!\cdot\!\partial z_k\,\right]\right]$$

(c)

In order to express the Lagrangian in terms of a quadratic operator we write

$$\begin{aligned} \bar{D}\bar{z}_j\!\cdot\! Dz_j \;&=\; \left(\partial_\mu\bar{z}_j - iA_\mu\bar{z}_j\right)\,D^\mu z_j \\ &=\; \partial_\mu\left(\bar{z}_j D^\mu z_j\right) \,-\, \bar{z}_j D_\mu D^\mu z_j \end{aligned}$$

Ignoring the total divergence we write the generating functional as

$$Z \,=\, \int \mathcal{D}A\mathcal{D}\lambda\;\exp\left[\frac{i}{g^2}\int d^2x\,\lambda\right]\int \mathcal{D}\bar{z}\mathcal{D}z\,\exp\left[\frac{i}{g^2}\int d^2x\,\bar{z}_j\left(-D^2 - \lambda\right)z_j\right]$$

Absorbing the factor of g^2 into the $N\!+\!1$ complex fields we write this in terms of a functional determinant.

$$\begin{aligned} Z \;&=\; \int \mathcal{D}A\mathcal{D}\lambda\;\exp\left[\frac{i}{g^2}\int d^2x\,\lambda\right]\,\left[\,\det\!\left(D^2+\lambda\right)\right]^{-(N+1)} \\ &=\; \int \mathcal{D}A\mathcal{D}\lambda\;\exp\left[-N\,\mathrm{tr}\ln\!\left(D^2+\lambda\right) \,+\, \frac{i}{g^2}\int d^2x\,\lambda\right] \end{aligned}$$

Where we are assuming that $N \gg 1$. We thus have the action

$$S\left[A,\lambda\right] \,=\, \int d^2x\,\left(\,i\,N\,\langle x|\ln\!\left(D^2+\lambda\right)|x\rangle \,+\, \frac{1}{g^2}\,\lambda\right)$$

Taking the variation with respect to A_μ

$$\frac{\delta S}{\delta A_\mu} \,=\, -2\,N\;\langle x|\left(D^2+\lambda\right)^{-1}D^\mu\,|x\rangle$$

If we take $A = 0$ and $\lambda = m^2$ this vanishes by translation invariance. Taking the variation with respect to λ and setting $\lambda = m^2$

$$\frac{\delta S}{\delta \lambda} \; = \; i\,N \; \langle x| \left(\partial^2 + m^2\right)^{-1} |x\rangle \; + \; \frac{1}{g^2}$$

Using dimensional regularization ($\epsilon = 2 - d$)

$$\langle x| \left(\partial^2 + m^2\right)^{-1} |x\rangle \; = \; -\int \frac{d^2p}{(2\pi)^2} \; \left(p^2 - m^2\right)^{-1} \; = \; \frac{i}{4\pi}\,\Gamma(\epsilon/2) \; \left(\frac{4\pi}{m^2}\right)^{\epsilon/2}$$

We write this in terms of a cutoff Λ as

$$\langle x| \left(\partial^2 + m^2\right)^{-1} |x\rangle \; = \; \frac{i}{2\pi} \; \ln\left(\frac{\Lambda}{m}\right)$$

We define a renormalized coupling g_r in terms of the mass scale M

$$\frac{1}{g^2} \; = \; \frac{1}{g_r^2} \; + \; \frac{N}{2\pi} \; \ln\left(\frac{\Lambda}{M}\right)$$

Thus we may write the solution to the λ equation of motion as

$$m \; = \; M \; \exp\left[-\,\frac{2\pi}{g_r^2 N}\right]$$

(d)

Expanding the action, the $\mathrm{O}(A^2)$ term is given by

$$S\left[A\right] \; \simeq \; N \int \frac{d^2k}{(2\pi)^2} \, \tilde{A}_\mu(k)\tilde{A}_\nu(-k) \left(\gamma\,\eta^{\mu\nu} - \tfrac{1}{2}\,\Gamma^{\mu\nu}(k)\right)$$

Where

$$A_\mu(x) \; = \; \int \frac{d^2k}{(2\pi)^2} \, \tilde{A}_\mu(k)\, e^{-ik\cdot x}$$

And, using dimensional regularization ($\epsilon = 2 - d$)

$$\gamma \; = \; i\int \frac{d^2p}{(2\pi)^2} \, \frac{1}{p^2 - m^2} \; = \; \frac{1}{4\pi}\,\Gamma(\epsilon/2) \; \left(\frac{4\pi}{m^2}\right)^{\epsilon/2}$$

And where

$$\Gamma^{\mu\nu}(k) \; = \; i\int \frac{d^2p}{(2\pi)^2} \, \frac{\left(2p^\mu - k^\mu\right)\left(2p^\nu - k^\nu\right)}{\left(p^2 - m^2\right)\left((p-k)^2 - m^2\right)}$$

Defining $q = p - xk$ we find

$$\Gamma^{\mu\nu}(k) \; = \; i\int_0^1 dx \int \frac{d^2q}{(2\pi)^2} \; \frac{4\,q^\mu q^\nu + (2x-1)^2\,k^\mu k^\nu}{\left(q^2 - \Delta\right)^2}$$

Where $\Delta = -x(1-x)k^2 + m^2$. Using

$$i\int \frac{d^2q}{(2\pi)^2} \; \frac{1}{\left(q^2 - \Delta\right)^2} \; = \; -\frac{1}{4\pi}\,(\epsilon/2)\Gamma(\epsilon/2)\,\frac{1}{\Delta}\,\left(\frac{4\pi}{\Delta}\right)^{\epsilon/2}$$

And

$$i\int \frac{d^2q}{(2\pi)^2} \; \frac{q^\mu q^\nu}{\left(q^2 - \Delta\right)^2} \; = \; \frac{1}{8\pi}\,\eta^{\mu\nu}\,\Gamma(\epsilon/2)\,\left(\frac{4\pi}{\Delta}\right)^{\epsilon/2}$$

We find the finite result

$$\gamma\,\eta^{\mu\nu} - \tfrac{1}{2}\,\Gamma^{\mu\nu}(k) \;=\; \frac{1}{4\pi}\int_0^1 dx\left[\eta^{\mu\nu}\,\ln\left(\frac{\Delta}{m^2}\right) + k^\mu k^\nu\,(2x-1)^2\,\frac{1}{2\Delta}\right]$$

This may be written as

$$\gamma\,\eta^{\mu\nu} - \tfrac{1}{2}\,\Gamma^{\mu\nu}(k) \;=\; \frac{1}{4\pi}\,\frac{1}{k^2}\left[k^2\,\eta^{\mu\nu}\,I_1(k^2/m^2) - k^\mu k^\nu\,I_2(k^2/m^2)\right]$$

Where,

$$I_1(a^2) \;=\; \int_0^1 dx\,\ln\left(1 - x(1-x)a^2\right)$$

And,

$$I_2(a^2) \;=\; -\frac{a^2}{2}\int_0^1 dx\,\frac{(2x-1)^2}{(1-x(1-x)a^2)}$$

It may be shown that $I_1(a^2) = I_2(a^2)$. Note that the quadratic term in $S[A]$, like the Maxwell action

$$S_{EM}[A] \;=\; -\frac{1}{4e^2}\int d^2x\,F_{\mu\nu}F^{\mu\nu} \;=\; -\frac{1}{2e^2}\int\frac{d^2k}{(2\pi)^2}\,\tilde{A}_\mu(k)\tilde{A}_\nu(-k)\left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right)$$

has a transverse propagator.