READING: Shankar, Chapter 1.

1. Vector spaces.

   (a) Let $\mathbf{V}$ be a linear vector space and let U, W be subspaces of $\mathbf{V}$ which are themselves vector spaces. Define the sum $U + W$ as the set of all $v \in \mathbf{V}$ that can be obtained as $u + w$ with $u \in U$ and $w \in W$. If the set of vectors in $\mathbf{V}$ that are in U and W simultaneously vanishes (that is if $U \cap W = 0$) then we say the sum $U + W$ is direct and write $U \oplus W$. Find a decomposition of the vector space R^4 into $U \oplus W$ for some U and W.

   (b) Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be two linear vector spaces with elements $|v\rangle^1$, $|v\rangle^2$. The tensor product space $\mathbf{V}_1 \otimes \mathbf{V}_1$ has elements

   $$|v^1 v^2\rangle \equiv |v\rangle^1 |v\rangle^2$$

   and we define this product of elements to be distributive with respect to the sum so that if

   $$|v\rangle^1 = \alpha|u\rangle^1 + \beta|w\rangle^1$$

   then

   $$|v^1 v^2\rangle = \alpha|u\rangle^1|v\rangle^2 + \beta|w\rangle^1|v\rangle^2$$

   and similarly for $|v\rangle^2$. Verify that this defines a new vector space.

   (c) If $\mathbf{V}_1$ and $\mathbf{V}_2$ have dimension n_1 and n_2 respectively what is the dimension of $\mathbf{V}_1 \otimes \mathbf{V}_2$?

   (d) Find a decomposition of $R^4 = \mathbf{V}_1 \otimes \mathbf{V}_2$ for some $\mathbf{V}_1$, $\mathbf{V}_2$.

2. The three Pauli matrices, σ_i, i=1,2,3 play an important role in the the theory of spin 1/2 particles. They are defined by

   $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   (a) Compute the commutator $[\sigma_i, \sigma_j] \equiv \sigma_i\sigma_j - \sigma_j\sigma_i$ and the anticommutator $\{\sigma_i, \sigma_j\} \equiv \sigma_i\sigma_j + \sigma_j\sigma_i$ for any two Pauli matrices.

   (b) Denote the two by two unit matrix by σ_0 and write σ_α, $\alpha = 0, 1, 2, 3$ for the identity matrix and the four Pauli matrices. Show that any two by two matrix M can be written in the form $M = \sum_\alpha a_\alpha \sigma_\alpha$ for some complex coefficients a_α. What are the conditions on a_α if M is to be unitary? What are the conditions for M Hermitian?

   (c) Consider the vector space consisting of all two by two complex matrices M. Show that this is an inner product space with the inner product of two matrices M, M' given by $\mathrm{Tr}(M' M^\dagger)$