Problem Set 2

Physics 330

Due October 12

Some abbreviations: A - Arfken & Weber, MW - Mathews & Walker.

1 Another ODE problem. MW 1-21.

The differential equation obeyed by the charge q on a capacitor C connected in series with a resistance R to a voltage

$$V = V_0 \left(\frac{t}{\tau}\right)^2 e^{-t/\tau} \tag{1}$$

is

$$R\frac{dq}{dt} + \frac{1}{C}q = V_0 \left(\frac{t}{\tau}\right)^2 e^{-t/\tau}. \tag{2}$$

Find $q(t)$ if $q(0) = 0$.

2. Fun with quaternions. Recall that we defined the quaternions in terms of real matrices I, J, K.

(i) Show that the definitions of I, J, K given in lecture satisfy the quaternion algebra:

$$I^2 = J^2 = K^2 = -1, \qquad IJ = K. \tag{3}$$

(ii) Show that the quaternion algebra is non-commutative by computing the commutator,

$$[q, q'] = qq' - q'q, \tag{4}$$

where q is a quaternion $q = \mathbf{1}q_1 + q_2 I + q_3 J + q_4 K$.

(iii) Show that for $\tilde{I} = \alpha_1 I + \alpha_2 J + \alpha_3 K$ to be a complex structure, we require that

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1.$$

(iv) Define the conjugate of q to be $\bar{q}$ where

$$\bar{q} = \mathbf{1}q_1 - q_2 I - q_3 J - q_4 K. \tag{5}$$

Note the analogy with z and $\bar{z}$. Show that $q\bar{q}$ is positive. The set of unit quaternions q satisfying,

$$q\bar{q} = 1,$$

are particular interesting. Show that the unit quaternions form a group by showing the following properties:

a. If q and q' are unit quaternions then qq' is a unit quaternion.

b. For every q, there is an inverse q^{-1} so that $qq^{-1} = q^{-1}q = 1$.

The other two requirements for a group are already clear. Namely, there is an identity element e satisfying $qe = eq = q$. The identity is just 1. The other requirement is associativity of the product rule

$$(qq')q'' = q(q'q'')$$

but this is a general property of quaternion (or matrix) multiplication.

(v) Let us connect the unit quaternions to some more familiar concepts. A unitary matrix is an $n \times n$ matrix M with complex entries satisfying,

$$M^{-1} = M^{\dagger},$$

where $M^{\dagger}$ is the complex-conjugate transposed matrix: $(M)^{\dagger}_{ab} = (M)^{*}_{ba}$.

Consider a 2×2 matrix,

$$M = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}.$$

If M is unitary with determinant one, what conditions must the z_i satisfy? Using these conditions, eliminate two of the z_i (say z_3 and z_4) and list the condition on the remaining two variables. Show that all unitary matrices M with determinant one form a group. The group of unitary determinant one matrices is the familiar group $SU(2)$. For $n \times n$ matrices, we get the group $SU(n)$.

We can now connect with our earlier discussion. Let us take

$$z_1 = q_1 + iq_2, \qquad z_2 = q_3 + iq_4.$$

Form a quaternion q by taking the combination $q = q_1 + q_2 I + q_3 J + q_4 K$. What condition must q obey so that the matrix M formed from z_1, z_2 is an element of $SU(2)$?

Some exercises with series. **3-6** Do MW 2-1, 2-6, 2-7, 2-11. For the first three problems, find the sums:

3. $1 + \frac{1}{4} - \frac{1}{16} - \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} - - + + \ldots.$

4. $1 + \frac{1}{9^2} + \frac{1}{25^2} + \frac{1}{49^2} + \ldots.$

5. $1 - \frac{1}{4^2} + \frac{1}{9^2} - \frac{1}{16^2} + - \ldots.$

6. Evaluate the series

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n^2 x^{2n-1}}{(2n-1)!} = x - \frac{4x^3}{3!} + \frac{9x^5}{5!} - + \ldots. \qquad (6)$$

Extra credit:

Show that the real function $x^{1+\frac{1}{n}}$ converges uniformly to x in the interval from 0 to 1. There are a number of ways to show this.