

### Problem Set 3

Physics 330

Due October 26

Some abbreviations: A - Arfken & Weber, MW - Mathews & Walker.

1. Another series problem. MW 2-8. Evaluate in closed form the sum

$$f(\theta) = \sin(\theta) + \frac{1}{3} \sin(2\theta) + \frac{1}{5} \sin(3\theta) + \frac{1}{7} \sin(4\theta) + \dots \quad (1)$$

You can assume that  $0 < \theta < \pi$  for definiteness.

2. We revisit the partial fraction representation of  $\text{ctn}$  using complex analytic techniques. Suppose that a function  $f(z)$  is analytic everywhere except at simple poles  $z = a_1, a_2, \dots$  with residues  $b_1, b_2, \dots$ . Further, as  $|z| \rightarrow \infty$ , assume that  $f$  is finite away from any poles.

(i) Show that,

$$f(z) = f(0) + \sum_n b_n \left( \frac{1}{z - a_n} + \frac{1}{a_n} \right). \quad (2)$$

(ii) Show that plugging in  $(\pi z)\text{ctn}(\pi z)$  gives the partial fraction expansion we derived in class.

3.

(i) Evaluate,

$$\int_0^{2\pi} d\theta \frac{1}{(a + b \cos^2 \theta)^2}. \quad (3)$$

(ii) For  $0 < s < 1$ , evaluate the integral:

$$\int_0^{2\pi} d\theta \frac{\cos^2(3\theta)}{1 - 2s \cos(2\theta) + s^2}. \quad (4)$$

(iii) Evaluate the integral,

$$\int_0^\infty dx \frac{(\ln x)^2}{1 + x^2}. \quad (5)$$

Be careful with branch cuts and the choice of the contour!

Some more exercises with complex analysis. **4-6** Do MW A-4, A-5, A-6.

4. The function  $f(z)$  has a simple pole of order  $n$  at  $z = z_0$ . Show that the function  $f'(z)/f(z)$  has a simple pole at  $z_0$ . What is the residue?

5. In the examples below,  $U(x, y)$  and  $V(x, y)$  denote the real and imaginary parts, respectively, of the analytic function  $W(z)$ . Find the missing quantities.

(a)  $U(x, y) = e^x \cos(y)$ ,  $V(x, y) = ?$ ,  $W(z) = ?$

(b)  $U(x, y) = ?$ ,  $V(x, y) = y(3x^2 - y^2 - 1)$ ,  $W(z) = ?$

(c)  $U(x, y) = ?$ ,  $V(x, y) = ?$ ,  $W(z) = \tan^{-1}z$ .

**6.** Find the residue of the function  $z^2 e^{1/\sin(z)}$  at the isolated (essential) singularity  $z = \pi$ .  
This is an interesting problem.