# Problem Set 5

Physics 330

Due November 16

Some abbreviations: A - Arfken & Weber, MW - Mathews & Walker.

**1.** For our first question this week, let us rederive the Cauchy principal value formula with a second choice of contour. Assume that the function $f$ is analytic except for isolated singularities. Further, assume that $f \to 0$ faster than $1/|z|$. If $f$ has a pole on the real axis at $x = x_o$, derive a formula for the principal value,

$$P \int_{-\infty}^{\infty} f(x)\, dx, \tag{1}$$

by closing the contour in the *lower* complex plane rather than the upper complex plane. Explain why the result is actually equivalent to the result obtained in class.

**2.** As a practice case, evaluate:

$$P \int_{0}^{\infty} \frac{x^{s-1}}{1-x}\, dx, \tag{2}$$

where $0 < s < 1$.

**3.** Let $f(z)$ be a function analytic inside a contour $C$ except at a finite number of poles. Assume that $f$ has no poles or zeroes on $C$.

(i) Find a formula for,

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}, \tag{3}$$

in terms of the number of zeroes and poles contained in $C$,

(ii) Use this result to give another proof that a polynomial of degree $n$ has $n$ roots.

These kinds of formulae have simple generalizations that are very useful for counting the number of zeroes or poles within some region without actually finding each pole or zero.

**4.** Suppose that $f(z)$ is analytic inside a closed contour. Show that $f$ attains its maximum value on the contour itself. This is a slight generalization of Liouville's theorem which we derived in class.

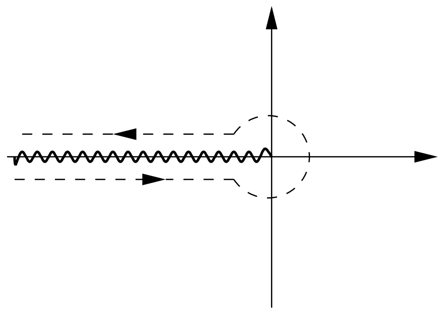

Figure 1: Integration contour for Problem 6.

**5.** Do M-W 1-33. Bessel's equation for $m = 0$ is

$$x^2 y'' + xy' + x^2 y = 0. \tag{4}$$

We found one solution

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - + \dots. \tag{5}$$

Show that a second solution exists of the form

$$J_0(x) \log(x) + Ax^2 + Bx^4 + Cx^6 + \dots \tag{6}$$

and find the first three coefficients $A, B, C$.

**6.** (A 11.1.16) Special functions often have integral representations. Show that the following definition of a Bessel function actually satisfies Bessel's equation by differentiation and substitution:

$$J_\nu(x) = \frac{1}{2\pi i} \int_C e^{(x/2)(w-1/w)} w^{-\nu-1} dw, \tag{7}$$

where $C$ is the contour drawn above.

Extra Credit: Due anytime before the end of the quarter. Solve M-W **A-6** analytically.