Problem Set 6

Physics 330

Due November 21

Some abbreviations: A - Arfken & Weber, MW - Mathews & Walker.

Since Thanksgiving is next week, this problem set is shorter than usual. Note the due date, and have a Happy Thanksgiving!

Practice with linear algebra **1-2**: do M-W 6-2, 6-3.

1. M-W 6-2: Let U be a unitary matrix and let x_1, x_2 be two eigenvectors of U belonging to eigenvalues λ_1, λ_2, respectively. Show that
(a) $|\lambda_1| = |\lambda_2| = 1$
(b) If $\lambda 1 \neq \lambda_2, \quad x_1^\dagger x_2 = 0$.

2. M-W 6-3: Suppose the matrices A and B are Hermitian and the matrices C and D are unitary. Prove that
(a) $C^{-1}AC$ is Hermitian
(b) $C^{-1}DC$ is unitary
(c) $i(AB - BA)$ is Hermitian.

3. We saw from the spectral theorem that any Hermitian matrix H can be diagonalized by a unitary matrix U,

$$H = H^\dagger, \qquad UU^\dagger = 1, \qquad UHU^\dagger = \text{ diagonal.} \tag{1}$$

We claimed in lecture that the set of unitarily diagonalizable matrices could be extended to include normal matrices N which are complex $n \times n$ matrices that satisfy,

$$N^\dagger N = NN^\dagger. \tag{2}$$

Show that this is true.

4. The final exercise is related to ideas that are important in modern theoretical physics, particularly gauge theory i.e. electromagnetism and its generalizations.

Suppose,

$$Q : V \to V \tag{3}$$

is a linear operator that satisfies $Q^2 = 0$.

(i) Show that $Im(Q)$ is a subspace of $Ker(Q)$. The quotient space $H^*(Q) = Ker(Q)/Im(Q)$ is called the 'cohomology of Q.'
(ii) Let,

$$Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

acting on $\mathbb{C}^2$. Compute $H^*(Q)$.

Suppose we have a chain of vector spaces $V_0, V_1, \ldots, V_n$ and linear operators

$$Q_i : V_i \to V_{i+1} \quad i = 0, 1, \ldots, n-1$$

so that:

(a) $Q_{i+1}Q_i = 0, \quad i = 0, \ldots, n-1,$
(b) $Ker(Q_0) = \{0\},$
(c) $Cok(Q_{n-1}) = \{0\}$, (Cok = Cokernel).

Define,

$$H^i(Q) \equiv Ker(Q_i)/Im(Q_{i-1}) \quad i = 1, 2, \ldots, n-1 \tag{4}$$
$$H^0(Q) \equiv Ker(Q_0) \tag{5}$$

(iii) Show that the vector spaces $H^i(Q)$ are well-defined.
(iv) Show that,

$$\sum_{p=0}^{n} t^p \dim V_p - \sum_{p=0}^{n-1} t^p \dim H^p(Q) = (1+t)Q(t)$$

where $Q(t)$ is a polynomial with non-negative coefficients.
(v) Show that,

$$\sum (-1)^i \dim V_i = \sum (-1)^i \dim H^i(Q).$$

This is called the Euler-Poincaré principle.