Problem Set 1: Solutions
Physics 330

Due Date: October 5, 2007

1. (A 8.2.6) Show that the substitution $u = y/x$ makes the equation $y' = g(y/x)$ separable.

This is fairly straightforward. If $y = ux$, then $y' = u'x + u$ and we have

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u - g(u) = 0$$
$$\frac{\mathrm{d}u}{u - g(u)} + \frac{\mathrm{d}x}{x} = 0$$

Thus, we have separated the equation, as desired. The formal solution can be found by integrating, yielding

$$x(u) = \mathcal{C} \int \frac{\mathrm{d}u}{g(u) - u},$$

inverting this equation to find $u(x)$, and substituting $u = y/x$ to yield $y(x)$. □

2. (A 8.2.17) Solve the first-order equation $y' + p(x)y = q(x)$ by assuming that $y(x) = u(x)v(x)$, where $v(x)$ is the solution to the homogeneous equation ($q = 0$).

We first solve for $v(x)$:

$$pv' + v = 0$$
$$\frac{\mathrm{d}v}{v} = -p(x)\mathrm{d}x$$
$$v(x) = \exp\left[-\int p(x)\mathrm{d}x\right]$$

Making the suggested substitution, we have

$$u'v + v'u + puv = q$$
$$\frac{u'}{u} + \frac{v'}{v} + p = \frac{q}{uv}$$

But since v is the homogeneous solution, we have $v' + pv = 0$. So the last two terms on the left-hand side cancel, and we have

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{q}{v}$$
$$\int \mathrm{d}u = u = \int \frac{q}{v}$$
$$y = v \int \frac{q}{v}$$

and our general solution is

$$\boxed{y(x) = \exp\left[-\int p(x)\mathrm{d}x\right]\left(\int q(x)\exp\left[\int p(x)\mathrm{d}x\right]\mathrm{d}x + \mathcal{C}\right)}$$

3. (A 8.6.25) Show that the particular solution to the differential equation $y'' + P(x)y' + Q(x)y = F(x)$ is given by

$$y_p = y_2 \int \mathrm{d}x \frac{y_1 F}{W} - y_1 \int \mathrm{d}x \frac{y_2 F}{W}$$

where $W = y_1 y_2' - y_1' y_2$ is the Wronskian and y_1 and y_2 are the homogeneous solutions to the equation.

We first show a result concerning the Wronskian. Taking its derivative with respect to x, we have

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}x} &= y_1 y_2'' - y_1'' y_2 \\ &= -y_1(P y_2' + Q y_2) + y_2(P y_1 + Q y_1) \\ &= -P(y_1 y_2' - y_1' y_2) = -PW\end{aligned}$$

and thus

$$W = \exp\left[-\int P\,\mathrm{d}x\right]$$

Now: similarly to the previous problem, we try a solution of the form $y_p(x) = y_1(x)u(x)$. Plugging this into the equation, we obtain

$$\begin{aligned}y_1''u + 2y_1'u' + y_1 u'' + P(y_1'u + y_1 u') + Qy_1 u = F \\ y_1 u'' + (2y_1' + Py_1)u' + (y_1'' + Py_1' + Qy_1)u = F\end{aligned}$$

Since y_1 is a homogeneous solution, the term proportional to u vanishes and we are left with a first-order differential equation in u':

$$\frac{\mathrm{d}u'}{\mathrm{d}x} + \left(\frac{2y_1'}{y_1} + P\right)u' = \frac{F}{y_1}$$

This equation can be solved by multiplying it by an integrating factor (see Eqn. (1-6) *et seq.* of Matthews & Walker):

$$\lambda(x) = \exp\left[\int \frac{2y_1'}{y_1} + P\mathrm{d}x\right] = \exp\left[2\ln y_1 + \int P\mathrm{d}x\right] = \frac{y_1^2}{W}$$

Multiplying our differential equation by $\lambda(x)$, we obtain

$$\begin{aligned}
\frac{y_1^2}{W}\mathrm{d}u' + \left(2y_1'y_1 + \frac{Py_1^2}{W}\right)\mathrm{d}x &= \frac{y_1F}{W}\mathrm{d}x \\
\frac{y_1^2}{W}\mathrm{d}u' + \left(2y_1'y_1 - \frac{W'y_1^2}{W^2}\right)\mathrm{d}x &= \frac{y_1F}{W}\mathrm{d}x \\
\mathrm{d}\left(\frac{y_1^2}{W}u'\right) &= \frac{y_1F}{W}\mathrm{d}x
\end{aligned}$$

and integrating this twice yields

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{W}{y_1^2}\int\frac{y_1F}{W} \\
u &= \int\left[\frac{W}{y_1^2}\int\frac{y_1F}{W}\right]
\end{aligned}$$

Since we have

$$\frac{W}{y_1^2} = \frac{y_2'}{y_1} - \frac{y_1'y_2}{y_1^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y_2}{y_1}\right),$$

our expression for u can be integrated by parts to yield

$$\begin{aligned}
u &= \frac{y_2}{y_1}\int\frac{y_1F}{W}\mathrm{d}x - \int\frac{y_2}{y_1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int\frac{y_1F}{W}\right)\mathrm{d}x \\
&= \frac{y_2}{y_1}\int\frac{y_1F}{W}\mathrm{d}x - \int\frac{y_2F}{W}\mathrm{d}x
\end{aligned}$$

and multiplying u by y_1 gives us our expression for y_p:

$$y_p = y_2\int\frac{y_1F}{W}\mathrm{d}x - y_1\int\frac{y_2F}{W}\mathrm{d}x$$

as desired. □

4. Show that $y'' = f(y)$ can be integrated immediately if both sides are multiplied by y'.

This technique is sometimes called the *method of quadrature*, and is fairly straightforward:

$$\begin{aligned}
y'y'' &= y'f(y) \\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}y'^2 &= \frac{\mathrm{d}}{\mathrm{d}x}f(y) \\
\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= 2f(y) + \mathcal{C}_1 \\
\frac{\mathrm{d}y}{\sqrt{2f(y) + \mathcal{C}_1}} &= \mathrm{d}x
\end{aligned}$$

$$\boxed{\int \frac{\mathrm{d}y}{\sqrt{2f(y)+\mathcal{C}_1}} = x + \mathcal{C}_2}$$

This is an implicit solution $x = f(y)$, which can (in principle, though not always in practice) be inverted to find $y(x)$.

5. (MW 1-1) Find the general solution of $x^2y' + y^2 = xyy'$.

Rearranging, we have

$$(x^2 - xy)\mathrm{d}y + y^2\mathrm{d}x = 0$$

This is a homogeneous differential equation, and so we make the substitution $y = xz$ to obtain

$$\begin{aligned}(x^2 - x^2z)(z\mathrm{d}x + x\mathrm{d}z) + x^2z^2\mathrm{d}x &= 0\\ z\mathrm{d}x &= (z-1)x\mathrm{d}z\\ \frac{\mathrm{d}x}{x} &= \left(1 - \frac{1}{z}\right)\mathrm{d}z\end{aligned}$$

Integrating out yields

$$\ln x + \mathcal{C} = z - \ln z$$

Substituting back in for y yields

$$\ln x + \mathcal{C} = \frac{y}{x} - \ln y + \ln x$$

$$\boxed{x = \frac{y}{\mathcal{C} + \ln y}}$$

I don't believe, however, that this relation can be simplified into a function $y(x)$ by use of elementary functions.

6. (MW 1-2) Find the general solution of $y' = \frac{x\sqrt{1+y^2}}{y\sqrt{1+x^2}}$.

This equation is immediately separable & integrable:

$$\begin{aligned}\frac{y\mathrm{d}y}{\sqrt{1+y^2}} &= \frac{x\mathrm{d}x}{\sqrt{1+x^2}}\\ \sqrt{1+y^2} &= \sqrt{1+x^2} + \mathcal{C}\\ 1 + y^2 &= 1 + x^2 + 2\mathcal{C}\sqrt{1+x^2} + \mathcal{C}^2\end{aligned}$$

$$\boxed{y = \pm\sqrt{x^2 + \mathcal{C}^2 + 2\mathcal{C}\sqrt{1+x^2}}}$$

Note that both positive and negative square roots are allowed, since the original equation is invariant under the substitution $y \to -y$.

7. (MW 1-3) Find the general solution of $y' = \frac{a^2}{(x+y)^2}$.

This equation cries out for the substitution $z = x + y$. Doing so yields

$$z' - 1 = \frac{a^2}{z^2}$$

which is separable:

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{a^2 + z^2}{z^2} \\ \frac{z^2 \mathrm{d}z}{z^2 + a^2} &= \mathrm{d}x \\ z - a \arctan \frac{z}{a} &= x + \mathcal{C} \end{aligned}$$

Substituting y back in gives

$$y - a \arctan \left(\frac{x+y}{a} \right) = \mathcal{C}$$

$$\boxed{x = a \tan \left(\frac{y + \mathcal{C}}{a} \right) - y}$$

Note that as in problem 5, this cannot be solved for y as a function of x. Them's the breaks.

8. (MW 1-4) Find the general solution of $y' + y \cos x = \frac{1}{2} \sin 2x$.

This is a linear first-order equation in y, and so can be solved with the use of integrating factors. Specifically, applying the notation of Matthews & Walker's Eqn. (1-6) *et seq.*, we have

$$\lambda(x) = \exp \left[\int \mathrm{d}x \cos x \right] = e^{\sin x}$$

and multiplying our equation by this factor on both sides should yield an exact differential:

$$\begin{aligned} e^{\sin x}(\mathrm{d}y + \cos x y \mathrm{d}x) &= \frac{1}{2} e^{\sin x} \sin 2x \mathrm{d}x \\ \mathrm{d}\left(y e^{\sin x} \right) &= e^{\sin x} \sin x \cos x \mathrm{d}x \\ y e^{\sin x} - \mathcal{C} &= \int \mathrm{d}x e^{\sin x} \sin x \cos x \\ &= e^{\sin x}(\sin x - 1) \end{aligned}$$

(The integral in the last step is most easily performed by making the substitution $z = \sin x$.) Thus,

$$\boxed{y = \sin x - 1 + \mathcal{C} e^{-\sin x}}$$

9. (MW 1-5) Find the general solution of $(1-x^2)y' - xy = xy^2$.

Rearranging, we have

$$(1-x^2)\mathrm{d}y = xy(y+1)\mathrm{d}x$$

which is separable:

$$\begin{aligned}
\frac{\mathrm{d}y}{y(y+1)} &= \frac{x\mathrm{d}x}{1-x^2} \\
\ln\left(\frac{y}{y+1}\right) &= -\frac{1}{2}\ln(|1-x^2|) + \mathcal{C} \\
\frac{y+1}{y} &= \mathcal{C}\sqrt{|1-x^2|}
\end{aligned}$$

$$\boxed{y = \frac{1}{\mathcal{C}\sqrt{|1-x^2|}-1}}$$

(Note that we effectively redefined $e^{-\mathcal{C}} \to \mathcal{C}$ in the third step. Hereafter, we will make such redefinitions without comment.)

10. (MW 1-6) Find the general solution of $2x^3y' = 1 + \sqrt{1+4x^2y}$.

In differential form, this becomes

$$2x^3\mathrm{d}y = (1+\sqrt{1+4x^2y})\mathrm{d}x.$$

This equation is isobaric if we assign x weight 1 and y weight -2. This suggests that we subsitute $y = v/x^2$:

$$\begin{aligned}
2x^3\left(\frac{\mathrm{d}v}{x^2} - \frac{2v\mathrm{d}x}{x^3}\right) &= (1+\sqrt{1+4v})\mathrm{d}x \\
2x\mathrm{d}v &= (1+4v+\sqrt{1+4v})\mathrm{d}x \\
\frac{\mathrm{d}v}{\sqrt{1+4v}(1+\sqrt{1+4v})} &= \frac{\mathrm{d}x}{2x} \\
\frac{1}{2}\ln(1+\sqrt{1+4v}) &= \frac{1}{2}\ln x + \mathcal{C}
\end{aligned}$$

Substituting back in yields

$$\sqrt{1+4x^2y} + 1 = 2\mathcal{C}x$$

and some algebra then gives us

$$\boxed{y = \mathcal{C}^2 - \frac{\mathcal{C}}{x}}$$

There is also an "isolated solution" of this differential equation, given by $y = -1/4x^2$. However, this isn't part of the general one-parameter family of functions that solve this differential equation.

11. (MW 1-7) Find the general solution of $y'' + y'^2 + 1 = 0$.

Since y itself doesn't appear in the equation, we can subsitute $z = y'$ and lower the order of the equation by one:

$$z' + z^2 + 1 = 0$$

$$\begin{aligned} \frac{\mathrm{d}z}{z^2+1} &= -\mathrm{d}x \\ \arctan z &= -x - \mathcal{C}_1 \\ z &= -\tan(x + \mathcal{C}_1) \end{aligned}$$

All that remains is to substitute $z = y'$ back in and integrate again:

$$\mathrm{d}y = -\tan(x + \mathcal{C}_1)\mathrm{d}x$$

$$\boxed{y = \ln(|\cos(x + \mathcal{C}_1)|) + \mathcal{C}_2}$$

12. (MW 1-8) Find the general solution of $y'' = e^y$.

This is easily solved by the method of quadrature, as in Problem 4:

$$\begin{aligned} y'y'' &= y'e^y \\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}y'^2 &= \frac{\mathrm{d}}{\mathrm{d}x}e^y \\ y'^2 &= 2e^y + \mathcal{C}_1 \end{aligned}$$

This can then be separated and integrated to yield

$$\int \frac{\mathrm{d}y}{\sqrt{2e^y + \mathcal{C}_1}} = x + \mathcal{C}_2$$

The integral over y is most easily done by making the substitution $z = \sqrt{2e^y + \mathcal{C}_1}$. This yields

$$\int \frac{2\mathrm{d}z}{z^2 - \mathcal{C}_1} = x + \mathcal{C}_2$$

This integral has a different functional form depending on whether $\mathcal{C}_1$ is positive, zero, or negative. If $\mathcal{C}_1 > 0$, then this becomes (noting that $z > \sqrt{\mathcal{C}_1}$ by definition)

$$\frac{2}{\sqrt{\mathcal{C}_1}}\operatorname{arccoth}\frac{z}{\sqrt{\mathcal{C}_1}} = x + \mathcal{C}_2$$

and rearranging yields

$$\begin{aligned} \sqrt{\frac{2e^y}{\mathcal{C}_1} + 1} &= \coth\left(\frac{\sqrt{\mathcal{C}_1}x}{2} + \mathcal{C}_2\right) \\ e^y &= \frac{\mathcal{C}_1}{2}\operatorname{csch}^2\left(\frac{\sqrt{\mathcal{C}_1}x}{2} + \mathcal{C}_2\right) \end{aligned}$$

If $\mathcal{C}_1 < 0$, then the integral becomes

$$\frac{2}{\sqrt{-\mathcal{C}_1}} \arctan \frac{z}{\sqrt{-\mathcal{C}_1}} = x + \mathcal{C}_2$$

and a similar rearrangement yields

$$e^y = -\frac{\mathcal{C}_1}{2} \sec^2 \left(\frac{\sqrt{-\mathcal{C}_1} x}{2} + \mathcal{C}_2 \right)$$

Finally, if $\mathcal{C}_1 = 0$, we have

$$e^{-y/2} = \mathcal{C}_2 - \frac{x}{\sqrt{2}}$$

In summary,

$$\boxed{y = \begin{cases} \ln\left(-\frac{\mathcal{C}_1}{2} \sec^2\left(\frac{\sqrt{-\mathcal{C}_1}x}{2} + \mathcal{C}_2\right)\right) & \mathcal{C}_1 < 0 \\ \ln\left(\left(\frac{x}{\sqrt{2}} + \mathcal{C}_2\right)^{-2}\right) & \mathcal{C}_1 = 0 \\ \ln\left(\frac{\mathcal{C}_1}{2} \operatorname{csch}^2\left(\frac{\sqrt{\mathcal{C}_1}x}{2} + \mathcal{C}_2\right)\right) & \mathcal{C}_1 > 0 \end{cases}}$$

13. (MW 1-9) Find the general solution of $x(1-x)y'' + 4y' + 2y = 0$.

This is a problem that's really best suited for the series method, since there doesn't seem to be anything else you can do with it. We begin with the ansatz

$$y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n$$

Note that the prefactor of x^α is required since $x = 0$ is a regular singular point of the original equation. Plugging this in, we have

$$x(1-x) \sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-2} \\ + 4 \sum_{n=0}^{\infty} a_n (n+\alpha) x^{n+\alpha-1} + 2 \sum_{n=0}^{\infty} a_n x^{n+\alpha} = 0$$

$$\sum_{n=0}^{\infty} \left[a_n (n+\alpha)(n+\alpha-1)(x^{n+\alpha-1} - x^{n+\alpha}) \right. \\ \left. + 4a_n (n+\alpha) x^{n+\alpha-1} + 2a_n x^(n+\alpha) \right] = 0$$

$$x^{\alpha-1} a_0 (\alpha(\alpha-1) + 4\alpha) + \sum_{n=0}^{\infty} [a_n(-(n+\alpha)(n+\alpha-1) + 2) \\ + a_{n+1}((n+\alpha+1)(n+\alpha) + 4(n+\alpha+1))] \, x^{n+\alpha} = 0$$

From the coefficient of the $x^{\alpha-1}$ equation, we see that we must have $\alpha = 0$ or $\alpha = -3$. We examine these two cases separately:

$\boldsymbol{\alpha = 0}$: In this case, the coefficient of the general term $x^{n+\alpha}$ becomes

$$a_n(-n(n-1)+2) + a_{n+1}((n+1)n + 4(n+1)) = 0$$

which simplifies to

$$a_{n+1} = \frac{n-2}{n+4} a_n.$$

We can see that this recursion relation will terminate at $n = 2$, and thus this solution will be a finite series. (This was by no means guaranteed.) Picking $a_0 = 1$, we find that $a_1 = -\frac{1}{2}$ and $a_2 = \frac{1}{10}$; so

$$y(x) = 1 - \frac{x}{2} + \frac{x^2}{10}.$$

$\boldsymbol{\alpha = -3}$: In this case, the coefficient of the general term $x^{n+\alpha}$ becomes

$$a_n(-(n-3)(n-4)+2) + a_{n+1}((n-2)(n-3) + 4(n-2))$$

which simplifies to

$$a_{n+1} = \frac{n-5}{n+1} a_n.$$

Again, the series terminates; going through the coefficients, we find that

$$y(x) = x^{-3}(1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5) = \frac{(1-x)^5}{x^3}$$

All told, the general solution is

$$\boxed{y = \mathcal{C}_1(x^2 - 5x + 10) + \mathcal{C}_2 \frac{(1-x)^5}{x^3}}$$

14. (MW 1-10) Find the general solution of $(1 + x^2)y^2 \mathrm{d}x - x^3 \mathrm{d}y = 0$.

This is obviously separable, and all that we have to do is integrate:

$$\int \frac{1-x}{x^3} \mathrm{d}x = \int \frac{\mathrm{d}y}{y^2}$$
$$-\frac{1}{2x^2} + \frac{1}{x} = -\frac{1}{y} + \mathcal{C}$$

$$\boxed{y = \frac{2x^2}{\mathcal{C}x^2 - 2x + 1}}$$