Problem Set 2: Solutions
Physics 330

Due Date: October 12, 2007

1. (MW 1-21) Find the solution to the differential equation

$$R\frac{\mathrm{d}q}{\mathrm{d}t}+\frac{1}{C}q=V_0\left(\frac{t}{\tau}\right)^2e^{-t/\tau}.$$

Since this is a linear equation, the general solution is of the form $q(t) = q_h(t)+q_p(t)$, where $q_p(t)$ is the particular solution and $q_h(t)$ is the homogeneous solution, i.e. q_h satisfies

$$R\frac{\mathrm{d}q_h}{\mathrm{d}t}+\frac{1}{C}q_h=0$$

Years of physics education have conditioned us to instantly write down the homogeneous solution to this equation:

$$q_h(t)=\mathcal{A}e^{-t/RC}$$

From Problem 2 of the last problem set, then, we know that

$$\begin{aligned}q_p(t)&=e^{-t/RC}\int e^{t/RC}\frac{V_0}{R}\left(\frac{t}{\tau}\right)^2e^{-t/\tau}\mathrm{d}t\\&=\frac{V_0}{R\tau^2}e^{-t/RC}\int t^2e^{t(1/RC-1/\tau)}\mathrm{d}t\end{aligned}$$

This integral has different forms depending on whether or not $\tau = RC$. If $\tau \neq RC$, then we have

$$\begin{aligned}q_p(t)&=\frac{V_0}{R\tau^2}e^{-t/RC}\left[\frac{e^{t(1/RC-1/\tau)}}{\frac{1}{RC}-\frac{1}{\tau}}\left(t^2-\frac{2t}{\frac{1}{RC}-\frac{1}{\tau}}+\frac{2}{\left(\frac{1}{RC}-\frac{1}{\tau}\right)^2}\right)\right]\\&=\frac{V_0\tau}{R}\frac{e^{-t/\tau}}{\frac{\tau}{RC}-1}\left(\frac{t^2}{\tau^2}-\frac{2t}{\tau\left(\frac{\tau}{RC}-1\right)}+\frac{2}{\left(\frac{\tau}{RC}-1\right)^2}\right)\end{aligned}$$

We can then apply the initial conditions to set $\mathcal{A}$:

$$0=q(0)=q_p(0)+q_h(0)=\frac{2V_0\tau}{R\left(\frac{\tau}{RC}-1\right)^3}+\mathcal{A}$$

$$\Rightarrow\mathcal{A}=-\frac{2V_0\tau}{R\left(\frac{\tau}{RC}-1\right)^3}$$

and so

$$q(t)=\frac{V_0\tau}{R}\frac{1}{\frac{\tau}{RC}-1}\left[\left(\frac{t^2}{\tau^2}-\frac{2t}{\tau\left(\frac{\tau}{RC}-1\right)}\right)e^{-t/\tau}+\frac{2}{\left(\frac{\tau}{RC}-1\right)^2}\left(e^{-t/\tau}-e^{-t/RC}\right)\right]$$

Meanwhile, if $\tau = RC$, we have instead

$$q_p(t) = \frac{V_0}{\tau^2 R} e^{-t/RC} \int t^2 \mathrm{d}t = \frac{V_0}{3\tau^2 R} \frac{t^3}{\tau^3} e^{-t/\tau}$$

and since $q_p(0) = 0$, we must have $\mathcal{A} = 0$ as well, i.e. $q(t) = q_p(t)$ for these initial conditions.

To summarize,

$$\boxed{q(t) = \begin{cases} \dfrac{V_0 \tau}{R} \dfrac{1}{\frac{\tau}{RC} - 1} \left[\left(\dfrac{t^2}{\tau^2} - \dfrac{2t}{\tau\left(\frac{\tau}{RC} - 1\right)} \right) e^{-t/\tau} \right. & \\ \qquad\qquad \left. + \dfrac{2}{\left(\frac{\tau}{RC} - 1\right)^2} \left(e^{-t/\tau} - e^{-t/RC} \right) \right] & \tau \neq RC \\ \dfrac{V_0}{3\tau^2 R} \dfrac{t^3}{\tau^3} e^{-t/\tau} & \tau = RC \end{cases}}$$

2. Fun with quaternions!

Throughout, we have

$$I = \begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix} \quad J = \begin{bmatrix} -i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{bmatrix} \quad K = \begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix}$$

(i) Show that $I^2 = J^2 = K^2 = -\mathbf{1}$ and $IJ = K$.
This is fairly straightforward:

$$I^2 = \begin{bmatrix} -\sigma^1\sigma^1 & 0 \\ 0 & -\sigma^1\sigma^1 \end{bmatrix} = \begin{bmatrix} -\mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{bmatrix} = -\mathbf{1}$$

$$J^2 = \begin{bmatrix} -\sigma^2\sigma^2 & 0 \\ 0 & -\sigma^2\sigma^2 \end{bmatrix} = \begin{bmatrix} -\mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{bmatrix} = -\mathbf{1}$$

$$K^2 = \begin{bmatrix} -\sigma^3\sigma^3 & 0 \\ 0 & -\sigma^3\sigma^3 \end{bmatrix} = \begin{bmatrix} -\mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{bmatrix} = -\mathbf{1}$$

$$IJ = \begin{bmatrix} 0 & -i\sigma^1\sigma^2 \\ i\sigma^1\sigma^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i(i\sigma^3) \\ i(i\sigma^3) & 0 \end{bmatrix} = K$$

as desired. ☐

(ii) Compute the quaternion commutator $[q, q']$ for two quaternions of the form $q = q_1\mathbf{1} + q_2 I + q_3 J + q_4 K$.
Let us compute the commutator of I and J using the algebraic relations derived above. We have

$$IJ = K \Rightarrow I^2 J = -J = IK \Rightarrow -JK = IK^2 = -I \Rightarrow J^2 K = -K = JI$$

and thus

$$[I,J] = 2K$$

Similarly, it can be shown that

$$[J,K] = 2I \qquad [K,I] = 2J$$

Finally, the matrix $\mathbf{1}$ commutes with I, J, and K, and every matrix commutes with itself. This greatly simplifies the calculation of the general commutator:

$$\begin{aligned}[q,q'] &= [q_1\mathbf{1} + q_2 I + q_3 J + q_4 K, q'_1\mathbf{1} + q'_2 I + q'_3 J + q'_4 K] \\ &= [q_2 I, q'_3 J] + [q_2 I, q'_4 K] + [q_3 J, q'_2 I] + [q_3 J, q'_4 K] \\ &\qquad\qquad\qquad + [q_4 K, q'_2 I] + [q_4 K, q'_3 J]\end{aligned}$$

$$\boxed{[q,q'] = 2(q_3 q'_4 - q_4 q'_3)I + 2(q_4 q'_2 - q_2 q'_4)J + 2(q_2 q'_3 - q_3 q'_2)K}$$

(iii) Show that for $\tilde{I} = \alpha_1 I + \alpha_2 J + \alpha_3 K$ to be a complex structure, we must have $\sum \alpha_i^2 = 1$.

This is fairly straightforward. If $\tilde{I}$ is a complex structure, we must have $\tilde{I}^2 = -\mathbf{1}$. This implies that

$$\begin{aligned}-\mathbf{1} &= (\alpha_1 I + \alpha_2 J + \alpha_3 K)(\alpha_1 I + \alpha_2 J + \alpha_3 K) \\ &= \alpha_1^2 I^2 + \alpha_2^2 J^2 + \alpha_3^2 K^2 \\ &\qquad + \alpha_1\alpha_2(IJ + JI) + \alpha_1\alpha_3(IK + KI) + \alpha_2\alpha_3(JK + KJ)\end{aligned}$$

The cross-terms vanish since these matrices anticommute with each other, and we're left with

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$$

as desired. ☐

(iv) Show that any quaternion multiplied by its conjugate ($\bar{q} = q_1\mathbf{1} - q_2 I - q_3 J - q_4 K$) is positive.

We just multiply it out:

$$\begin{aligned}q\bar{q} &= (q_1\mathbf{1} + q_2 I + q_3 J + q_4 K)(q_1\mathbf{1} - q_2 I - q_3 J - q_4 K) \\ &= q_1^2\mathbf{1} - q_2^2 I^2 - q_3^2 J^2 - q_4^2 K^2 + q_1(q_2 I + q_3 J + q_4 K - q_2 I - q_3 J - q_4 K) \\ &\qquad - q_2 q_3(IJ + JI) - q_2 q_4(IK + KI) - q_3 q_4(JK + KJ) \\ &= (q_1^2 + q_2^2 + q_3^2 + q_4^2)\mathbf{1}\end{aligned}$$

which is a positive multiple of the identity, as desired.

a. Show that if q and q' are unit quaternions, so is qq'.

We first show something more general: for any two quaternions q and q', $\overline{qq'} = \bar{q}'\bar{q}$. Proof: multiply it out.

$$\begin{aligned}&(q_1\mathbf{1} + q_2 I + q_3 J + q_4 K)(q'_1\mathbf{1} + q'_2 I + q'_3 J + q'_4 K) \\ &\quad = (q_1 q'_1 - q_2 q'_2 - q_3 q'_3 - q_4 q'_4)\mathbf{1} + (q_1 q'_2 + q_2 q'_1 + q_3 q'_4 - q_4 q'_3)I \\ &\quad\; + (q_1 q'_3 + q_3 q'_1 + q_4 q'_2 - q_2 q'_4)J + (q_1 q'_4 + q_4 q'_1 + q_2 q'_3 - q_3 q'_2)K\end{aligned}$$

while

$$\begin{aligned}&(q_1'\mathbf{1} - q_2'I - q_3'J - q_4'K)(q_1\mathbf{1} - q_2I - q_3J - q_4K)\\&\quad = (q_1q_1' - q_2q_2' - q_3q_3' - q_4q_4')\mathbf{1} - (q_1q_2' + q_2q_1' + q_3q_4' - q_4q_3')I\\&\qquad - (q_1q_3' + q_3q_1' + q_4q_2' - q_2q_4')J - (q_1q_4' + q_4q_1' + q_2q_3' - q_3q_2')K\end{aligned}$$

These are obviously conjugates of each other.
Now that we've established that, proving the closure of the unit quaternion group is trivial:

$$qq'\overline{qq'} = (qq')(\bar{q}'\bar{q}) = q(q'\bar{q}')\bar{q} = q\bar{q} = \mathbf{1}$$

as desired.

b. Show that every unit quaternion has an inverse.
We know that $q\bar{q} = \mathbf{1}$. Multiplying by $\bar{q}$ on the left yields $\bar{q}(q\bar{q}) = (\bar{q}q)\bar{q} = \bar{q}$. But since the identity element is unique, we must have $\bar{q}q = \mathbf{1}$. Thus, $q\bar{q} = \bar{q}q = \mathbf{1}$, and so $q^{-1} = \bar{q}$.

(v) Connect the above discussion to the unitary group $SU(2)$.

- If a unitary matrix M has determinant one, what conditions must its entries satisfy?
  If $|M| = 1$, then the equation $M^{-1} = M^\dagger$ becomes

  $$\begin{bmatrix} z_4 & -z_2 \\ -z_3 & z_1 \end{bmatrix} = \begin{bmatrix} z_1^* & z_3^* \\ z_2^* & z_4^* \end{bmatrix}$$

  This implies that we must have

  $$\boxed{z_3 = -z_2^* \qquad z_4 = z_1^*}$$

  However, we can't arbitrarily pick z_1 and z_2; to have determinant one, we must also have

  $$z_1z_4 - z_2z_3 = 1$$

  In terms of z_1 and z_2, this becomes

  $$\boxed{|z_1|^2 + |z_2|^2 = 1}$$

- Show that all unitary matrices with determinant one form a group.
  We need to check the four group axioms:
  - Suppose M and N are unitary matrices with determinant one. Then MN is unitary, since $(MN)^{-1} = N^{-1}M^{-1} = N^\dagger M^\dagger = (N^TM^T)^* = ((MN)^T)^* = (MN)^\dagger$; and MN has determinant one, since $\det(MN) = \det(M)\det(N) = 1$. Thus, the unitary matrices with determinant one are closed under multiplication.

- Suppose M is a unitary matrix with determinant one. Then $M^{-1} = M^\dagger$ is a unitary matrix, since $(M^{-1})^{-1} = M = (M^\dagger)^\dagger = (M^{-1})^\dagger$; and M^{-1} has determinant one, since $\det(M^{-1}) = \det(M)^{-1} = 1$. Thus, the unitary matrices with determinant one are closed under inversion.
- The identity element of this group is the 2×2 identity matrix, since $\det(\mathbf{1}) = 1$ and $\mathbf{1}^\dagger = \mathbf{1}^{-1} = \mathbf{1}$.
- The product rule for unitary matrices with determinant one is associative, since matrix multiplication is associative.

Thus, the unitary matrices with determinant one form a group. □

- Let $z_1 = q_1 + iq_2$ and $z_2 = q_3 + iq_4$. Let q be the quaternion defined by $q = q_1\mathbf{1} + q_2 I + q_3 J + q_4 K$. What condition must q obey so that the matrix M formed from z_1 and z_2 is in $SU(2)$?
  If the matrix formed from q is to be in $SU(2)$, we must have

$$|z_1|^2 + |z_2|^2 = 1 \quad \Rightarrow \quad q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

So under this rule, the unit quaternions have a one-to-one correspondence with the matrices of $SU(2)$. As it happens, this can be shown to be a *group isomorphism* between the unit quaternions and $SU(2)$; if ϕ is the above invertible map from the unit quaternions to the unitary matrices, then $\phi(qq') = \phi(q)\phi(q')$ for all unit quaternions q, q'.

3. (MW 2-1) Find the sum of the series $S = 1 + \frac{1}{4} - \frac{1}{16} - \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} - \dots$.

Adding together successive pairs of terms gives

$$S = \frac{5}{4} - \frac{5}{64} + \frac{5}{1024} - \dots = \frac{5}{4}\sum_{n=0}^{\infty}\left(-\frac{1}{16}\right)^n$$

Thus,

$$\boxed{S = \frac{5}{4} \times \frac{1}{1 + \frac{1}{16}} = \frac{20}{17}}$$

4. (MW 2-6) Find the sum of the series $S = 1 + \frac{1}{9^2} + \frac{1}{25^2} + \frac{1}{49^2} + \dots$.

We can rewrite this as

$$S = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}$$

This is just the odd terms of a Riemann zeta function; in other words,

$$\begin{aligned}S &= 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots - \frac{1}{2^4} - \frac{1}{4^4} - \frac{1}{6^4} - \dots \\ &= \sum_{n=0}^{\infty} \frac{1}{n^4} - \sum_{n=0}^{\infty} \frac{1}{(2n)^4} \\ &= \left(1 - \frac{1}{16}\right) \sum_{n=0}^{\infty} \frac{1}{n^4} \\ &= \frac{15}{16}\zeta(4)\end{aligned}$$

$$\boxed{S = \frac{\pi^4}{96}}$$

5. (MW 2-7) Find the sum of the series $S = 1 - \frac{1}{4^2} + \frac{1}{9^2} - \frac{1}{16^2} + \dots$.

The method to solve this problem is almost identical to that used for the previous one:

$$\begin{aligned}S &= 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots - 2\left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots\right) \\ &= \zeta(4) - \frac{2}{2^4}\zeta(4) = \frac{7}{8}\zeta(4)\end{aligned}$$

$$\boxed{S = \frac{7\pi^4}{720}}$$

6. (MW 2-11) Evaluate the series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n^2 x^{2n-1}}{(2n-1)!} = x - \frac{4x^3}{3!} + \frac{9x^5}{5!} - \dots$$

in closed form.

We have

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}$$

If we differentiate $\sin x$, we get

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)x^{2n-2}}{(2n-1)!}$$

$$-\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)(2n-2)x^{2n-3}}{(2n-1)!}$$

We can multiply these series by the appropriate powers of x to yield

$$x\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)x^{2n-1}}{(2n-1)!}$$

$$-x^2\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)(2n-2)x^{2n-1}}{(2n-1)!}$$

Suppose we now take a linear combination of $\sin x$, $x\cos x$, and $-x^2\sin x$. This would yield the series

$$\begin{aligned} &-\mathcal{A}x^2\sin x + \mathcal{B}x\cos x + \mathcal{C}\sin x \\ &\qquad = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!}\left(\mathcal{A}(2n-1)(2n-2) + \mathcal{B}(2n-1) + \mathcal{C}\right) \end{aligned}$$

But

$$\frac{1}{4}(2n-1)(2n-2) + \frac{3}{4}(2n-1) + \frac{1}{4} = n^2$$

So to get the desired series in closed form, we set $\mathcal{A} = \frac{1}{4}$, $\mathcal{B} = \frac{3}{4}$, and $\mathcal{C} = \frac{1}{4}$. Thus,

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}n^2x^{2n-1}}{(2n-1)!} = \frac{1}{4}(1-x^2)\sin x + \frac{3}{4}x\cos x}$$

Extra Credit: Show that the sequence of functions $f_n(x) = x^{1+\frac{1}{n}}$ converges uniformly to $f(x) = x$ on the interval $[0,1]$.

The definition of uniform convergence is thus: for a given ϵ, we must be able to find some number N such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in [0,1]$ and all $n > N$. We will explicitly construct an N satisfying this condition.

Let $g_n(x) = f(x) - f_n(x) = x(1 - x^{\frac{1}{n}})$. This function vanishes at $x = 0$ and $x = 1$, but is non-zero elsewhere. Taking the derivative of $g_n(x)$, we find that

$$g_n'(x) = 1 - \left(1 + \frac{1}{n}\right)x^{\frac{1}{n}}$$

If we set this equal to zero, we find that $g(x)$ only has one extremum on the interval $[0,1]$, at

$$x_n = \left(\frac{n}{n+1}\right)^n,$$

and at this point, $g(x)$ has the value

$$g_n(x_n) = \left(\frac{n}{n+1}\right)^n\left(1 - \frac{n}{n+1}\right) = \frac{1}{n}\left(\frac{n}{n+1}\right)^{n+1}$$

This will be the value at which $|g_n(x)| = |f_n(x) - f(x)|$ is greatest; so if we choose our N such that $|g_N(x_N)| < \epsilon$, we have proved uniform convergence. We can do this by picking N such that $N > \frac{1}{\epsilon}$; then for all $n > N$, we have

$$|g_n(x_n)| = \frac{1}{n}\left(\frac{n}{n+1}\right)^{n+1} < \frac{1}{n} < \frac{1}{N} < \epsilon$$

as desired. Thus, the sequence of functions $f_n(x)$ converges uniformly to x on the interval $[0, 1]$. ☐