Problem Set 3: Solutions
Physics 330

Due Date: October 26, 2007

1. (MW 2-8) Evaluate in closed form the sum

$$f(\theta) = \sin\theta + \frac{1}{3}\sin 2\theta + \frac{1}{5}\sin 3\theta + \dots$$

where $0 < \theta < \pi$.

We can write this as the imaginary part of a series involving $e^{i\theta}$:

$$f(\theta) = \Im\left[e^{i\theta} + \frac{1}{3}e^{2i\theta} + \frac{1}{5}e^{3i\theta} + \dots\right]$$

The rest is just manipulation:

$$\begin{aligned}
f(\theta) &= \frac{1}{2}\Im\Big[e^{i\theta} + \frac{1}{2}e^{3i\theta/2} + \frac{1}{3}e^{2i\theta} + \frac{1}{4}e^{5i\theta/2} + \dots \\
&\qquad\qquad + e^{i\theta} - \frac{1}{2}e^{3i\theta/2} + \frac{1}{3}e^{2i\theta} - \frac{1}{4}e^{5i\theta/2} + \dots\Big] \\
&= \frac{1}{2}\Im\Big[e^{i\theta/2}\Big(e^{i\theta/2} + \frac{1}{2}e^{i\theta} + \frac{1}{3}e^{3i\theta/2} + \frac{1}{4}e^{2i\theta} + \dots \\
&\qquad\qquad + e^{i\theta/2} - \frac{1}{2}e^{i\theta} + \frac{1}{3}e^{3i\theta/2} - \frac{1}{4}e^{2i\theta} + \dots\Big)\Big] \\
&= \frac{1}{2}\Im\left[e^{i\theta/2}\left(-\ln\left(1 - e^{i\theta/2}\right) + \ln\left(1 + e^{i\theta/2}\right)\right)\right]
\end{aligned}$$

We now wish to write $1 + e^{i\theta/2}$ and $1 - e^{i\theta}$ in terms of a magnitude and a phase. Finding the magnitude can be done either in terms of trigonometry or

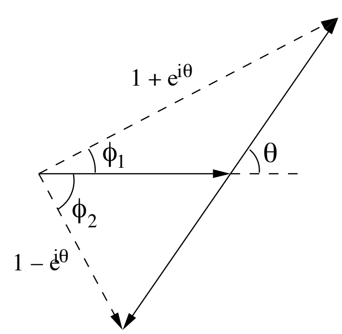

Figure 1: Phasors for $1 + e^{i\theta}$ and $1 - e^{i\theta}$.

by multiplying by the complex conjugate:

$$\left|1 + e^{i\theta/2}\right|^2 = (1 + e^{i\theta/2})(1 + e^{-i\theta/2}) = 2\left(1 + \cos\frac{\theta}{2}\right)$$

$$\left|1 - e^{i\theta/2}\right|^2 = (1 - e^{i\theta/2})(1 - e^{-i\theta/2}) = 2\left(1 - \cos\frac{\theta}{2}\right)$$

Denoting the phases of $1 + e^{i\theta/2}$ and $1 - e^{i\theta./2}$ by ϕ_1 and $-\phi_2$, respectively (see Figure 1), we then have

$$\begin{aligned}
&\ln\left(1 + e^{\imath\theta}/2\right) - \ln\left(1 - e^{i\theta/2}\right) \\
&\qquad = \ln\left(\sqrt{2(1 + \cos\theta/2)}\right) + i\phi_1 - \ln\left(\sqrt{2(1 - \cos\theta/2)}\right) - i\phi_2 \\
&\qquad = \frac{1}{2}\ln\left(\frac{1 + \cos\frac{\theta}{2}}{1 - \cos\frac{\theta}{2}}\right) + i(\phi_1 + \phi_2)
\end{aligned}$$

But from the geometry of Figure 1, we see that $\phi_1 + \phi_2 = \frac{\pi}{2}$ regardless of θ; and applying a standard trigonometric identity to the argument of the logarithm (valid when $0 < \frac{\theta}{2} < \frac{\pi}{2}$, we obtain

$$\ln\left(1 + e^{\imath\theta}/2\right) - \ln\left(1 - e^{i\theta/2}\right) = -\frac{1}{2}\ln\left(\tan^2\frac{\theta}{4}\right) + \frac{i\pi}{2}$$

Thus, we have

$$f(\theta) = \frac{1}{2}\Im\left[\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\left(-\ln\left(\tan\frac{\theta}{4}\right) + \frac{i\pi}{2}\right)\right]$$

$$\boxed{f(\theta) = \frac{\pi}{4}\cos\frac{\theta}{2} - \frac{1}{2}\sin\frac{\theta}{2}\ln\left(\tan\frac{\theta}{4}\right)}$$

Note: One might be somewhat suspicious of the resummation of the two series above: the series used for $\ln(1 \pm z)$ are only guaranteed to converge when $|z| < 1$, whereas we are concerned with the case where $|z| = 1$. As it happens, there is a general result due to Weierstrass that applies to this case: Suppose that we have a series $\sum_n a_n z^n$ which converges everywhere within the ball $|z| < 1$. Suppose further that the ratio of two successive terms can be written as

$$\frac{a_n}{a_{n-1}} = 1 - \frac{s}{n} + \frac{A(n)}{n^\lambda}$$

where s is a complex constant, $A(n)$ is a bounded function of n and $\lambda > 1$. Then it can be shown that:

- If $\Re(s) > 1$, the series converges absolutely for $|z| = 1$;
- If $\Re(s) \leq 0$, the series diverges for $|z| = 1$; and

- If $0 < \Re(s) \leq 1$, the series converges for $|z| = 1$ with the exception of the point $z = 1$, where it diverges.

In our case, we have

$$\frac{a_n}{a_{n-1}} = 1 - \frac{1}{n}$$

and thus the third case applies; since we are not concerned with the case $\theta = 0$, we have $e^{i\theta/2} \neq 0$, and the series does indeed converge for all θ under consideration.

2. Let $f(z)$ be an function which is complex-valued everywhere except at a series of simple poles $a_1, a_2, a_3, \ldots$, with residues $b_1, b_2, b_3, \ldots$ respectively, and with $f(z)$ finite as $|z| \to \infty$.

(i) Show that

$$f(z) = f(0) + \sum_n b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right).$$

Define a new function $g(z)$ such that

$$g(z) = f(z) - \sum_n \frac{b_n}{z - a_n}$$

This function is regular everywhere, since $f(z)$ only has simple poles and the residues at the points a_n are now zero. (In essence, $g(z)$ is $f(z)$ with the singularities subtracted from it.) Since $f(z)$ is finite as $|z| \to \infty$, and

$$\lim_{|z| \to \infty} \frac{b_n}{z - a_n} = 0,$$

we conclude that $g(z)$ is also finite as $|z| \to \infty$. Thus, $g(z)$ is bounded, since it is regular everywhere and does not diverge at infinity. We conclude from Liouville's theorem that

$$g(z) = C$$

where C is a constant, and thus

$$f(z) = \sum_n \frac{b_n}{z - a_n} + C.$$

To determine C, we set $z = 0$ in the above equation (assuming that $a_n \neq 0$ for all n), and obtain

$$f(0) + \sum_n \frac{b_n}{a_n} = C$$

Thus,

$$f(z) = f(0) + \sum_n b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right)$$

as desired. □

(ii) Using this result, find a partial fraction expansion for $f(z) = \pi z \cot(\pi z)$.

We first note that

$$\lim_{z \to 0} \pi z \cot(\pi z) = 1$$

(use L'Hôpital's rule to show this.) Thus, the singularity at $z = 0$ is removable, and we have

$$\cot(\pi z) = \frac{1}{\pi z} + \mathcal{O}(z) + \dots$$

We still need to know the locations and residues of the other poles of $f(z)$. The locations are easy: we will have a pole whenever $\tan(\pi z) = 0$, i.e. at all integers (except, as was shown above, zero.) To find the residue at a given integer n, we let $z' = z - n$ and use the above Laurent series for $\cot(\pi z)$:

$$\begin{aligned} f(z) = f(z'+n) &= \pi(z'+n)\cot(\pi(z'+n)) \\ &= \pi z' \cot(\pi z') + \pi n \cot(\pi z') \quad (\text{since } \cot x = \cot(x + n\pi)) \\ &= 1 + \mathcal{O}(z'^2) + \dots + \pi n \left(\frac{1}{\pi z'} + \mathcal{O}(z') + \dots \right) \\ &= \frac{n}{z'} + 1 + \mathcal{O}(z') + \dots \end{aligned}$$

and thus the residue at $z = n$ is n.

All that remains is to apply the result from part (i):

$$\begin{aligned} \pi z \cot(\pi z) &= 1 + \sum_{n \neq 0} n \left(\frac{1}{z-n} + \frac{1}{n} \right) \\ &= 1 + \sum_{n \neq 0} \frac{z}{z-n} \\ &= 1 + \sum_{n=1}^{\infty} \left(\frac{z}{z-n} + \frac{z}{z+n} \right) \\ &= 1 + \sum_{n=1}^{\infty} \frac{2z^2}{z^2 - n^2} \end{aligned}$$

$$\boxed{\pi z \cot(\pi z) = 1 + 2z^2 \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}}$$

as desired. □

3. (i) Evaluate the integral

$$I(a,b) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{(a + b\cos^2\theta)^2}$$

If $b = 0$, the integral is trivial; so we assume that $b \neq 0$. Moreover, we can also take $b > 0$, since $I(a, b) = I(-a, -b)$. Finally, we note that the denominator of the integrand will go to zero for some θ if $-b \leq a \leq 0$; thus, we exclude these cases from consideration, since $I(a, b)$ is divergent. With this in mind: We have

$$
\begin{aligned}
I(a,b) &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{(a + b\cos^2\theta)^2} \\
&= \int_0^{2\pi} \frac{\mathrm{d}\theta}{(a + \frac{b}{2}(1 + \cos 2\theta))^2} \\
&= \frac{4}{b^2} \int_0^{2\pi} \frac{\mathrm{d}\theta}{(1 + \frac{2a}{b} + \cos 2\theta)^2} \\
&= \frac{2}{b^2} \int_0^{4\pi} \frac{\mathrm{d}\theta'}{(1 + \frac{2a}{b} + \cos\theta')^2} \qquad \text{(setting } \theta' = 2\theta) \\
&= \frac{4}{b^2} \int_0^{2\pi} \frac{\mathrm{d}\theta'}{(1 + \frac{2a}{b} + \cos\theta')^2} \qquad \text{(by periodicity of } \cos\theta'.)
\end{aligned}
$$

Now let $z = e^{i\theta'}$. As θ' goes from 0 to 2π, z will traverse the unit circle in the complex plane, and so we can apply the Cauchy Residue Theorem to evaluate this integral. Noting that $\mathrm{d}\theta' = \frac{\mathrm{d}z}{iz}$ and $\cos\theta' = \frac{1}{2}(z + z^{-1})$, this becomes

$$
\begin{aligned}
I(a,b) &= \frac{4}{b^2} \oint_C \frac{\mathrm{d}z}{iz} \frac{1}{(1 + \frac{2a}{b} + \frac{1}{2}(z + z^{-1}))^2} \\
&= \frac{16}{ib^2} \oint_C \frac{z\,\mathrm{d}z}{(z^2 + 2(1 + \frac{2a}{b})z + 1)^2} \\
&= \frac{16}{ib^2} \oint_C \frac{z\,\mathrm{d}z}{(z - z_+)^2 (z - z_-)^2},
\end{aligned}
$$

where $z_\pm$ are the roots of the polynomial $z^2 + 2(1 + \frac{2a}{b})z + 1$:

$$
z_\pm = -1 - \frac{2a}{b} \pm \sqrt{\left(1 + \frac{2a}{b}\right)^2 - 1}
$$

We see that the integrand has poles of order two at $z = z_\pm$; the question becomes which of these are inside the unit circle (i.e. our contour.) We examine the cases for $a > 0$ and $a < -b$ separately:

$\boldsymbol{a > 0}$: In this case, we have $-1 < z_+ < 0$ and $z_- < -1$; so only the pole at z_+ is enclosed in our contour. Since this pole has order two, its residue

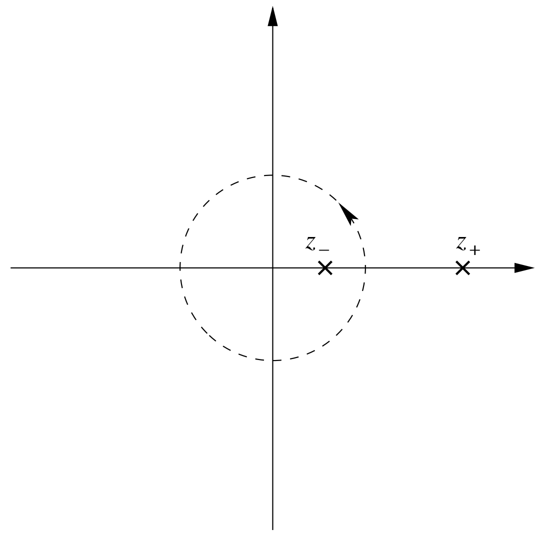

Figure 2: Integration contour and poles for Problem 3(i).

is given by

$$\begin{aligned}
\mathrm{Res}(f(z))_{z=z_+} &= \frac{\mathrm{d}}{\mathrm{d}z}\left[f(z)(z-z_+)^2\right]_{z=z_+} \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z}{(z-z_-)^2}\right]_{z=z_+} \\
&= -\frac{z_+ + z_-}{(z_+ - z_-)^3} \\
&= \frac{b^2(2a+b)}{32(a^2+ab)^{3/2}}
\end{aligned}$$

$\boldsymbol{a < -b}$: In this case, $0 < z_- < 1$ and $z_+ > 1$; so the residue at z_- is the only one that concerns us:

$$\begin{aligned}
\mathrm{Res}(f(z))_{z=z_-} &= \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z}{(z-z_-)^2}\right]_{z=z_+} \\
&= \frac{z_+ + z_-}{(z_+ - z_-)^3} \\
&= -\frac{b^2(2a+b)}{32(a^2+ab)^{3/2}}
\end{aligned}$$

So the final result is simply

$$I(a,b) = \frac{16}{ib^2} \times 2\pi i \times \begin{cases} \frac{b^2(2a+b)}{32(a^2+ab)^{3/2}} & a > 0 \\ -\frac{b^2(2a+b)}{32(a^2+ab)^{3/2}} & a < -b \end{cases}$$

$$\boxed{I(a,b) = \pi\frac{2a+b}{(a^2+ab)^{3/2}} \times \mathrm{sign}(a)}$$

(under the assumptions made at the start of the problem.)

(ii) Evaluate the integral

$$I(s) = \int_0^{2\pi} \mathrm{d}\theta \frac{\cos^2(3\theta)}{1 - 2s\cos 2\theta + s^2}$$

where $s \in (0,1)$.

We use the same tricks here that we did in the last part to set up the problem:

$$\begin{aligned} I(s) &= \frac{1}{2}\int_0^{2\pi} \mathrm{d}\theta \frac{1+\cos 6\theta}{1 - 2s\cos 2\theta + s^2} \\ &= \int_0^{\pi} \mathrm{d}\theta \frac{1+\cos 6\theta}{1 - 2s\cos 2\theta + s^2} \quad \text{(periodicity again)} \end{aligned}$$

Letting $z = e^{2i\theta}$, we have $\mathrm{d}\theta = \frac{\mathrm{d}z}{2iz}$, $\cos 2\theta = \frac{1}{2}(z + z^{-1})$, and $\cos 6\theta = \frac{1}{2}(z^3 + z^{-3})$; so this becomes

$$\begin{aligned} I(s) &= \oint_C \frac{\mathrm{d}z}{2iz} \frac{1 + \frac{1}{2}(z^3 + z^{-3})}{1 - s(z + z^{-1}) + s^2} \\ &= \frac{1}{-4is}\oint_C \mathrm{d}z \frac{2 + z^3 + z^{-3}}{z^2 - (s + s^{-1})z + 1} \\ &= -\frac{1}{4is}\oint_C \mathrm{d}z \frac{(z^3+1)^2}{z^3(z-s)(z-s^{-1})} \end{aligned}$$

where C is the unit circle again.

This integrand has poles of order one at s and $\frac{1}{s}$, and a pole of order three at zero. Since $0 < s < 1$, the pole at $\frac{1}{s}$ will be outside our contour; the other two poles will always be inside our contour.

It remains to find the two residues. At $z = s$, the residue is simply

$$\begin{aligned} \mathrm{Res}(f(z))_{z=s} &= (z-s)f(z)|_{z=s} \\ &= \frac{(s^3+1)^2}{s^2(s^2-1)} \end{aligned}$$

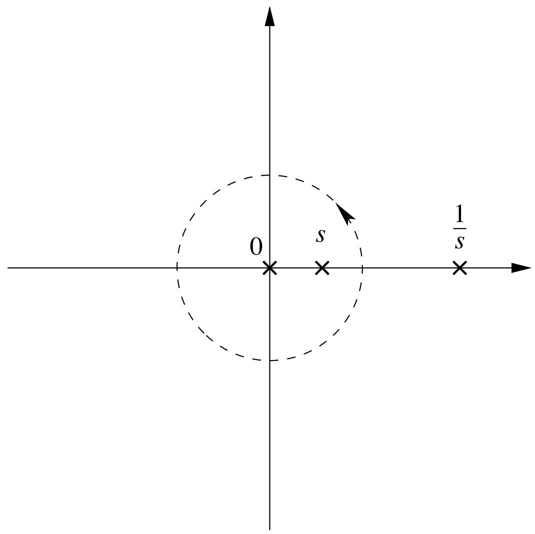

Figure 3: Integration contour and poles for Problem 3(ii).

The residue at $z = 0$ is a little trickier to find:

$$\begin{aligned}
\operatorname{Res}(f(z))_{z=0} &= \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[\frac{(z^3+1)^2}{(z-s)(z-s^{-1})}\right]_{z=0} \\
&= \frac{1}{2}\left[\frac{30z^4+12z}{(z-s)(z-s^{-1})} + 2\frac{(z^3+1)^2}{(z-s)^3(z-s^{-1})}\right. \\
&\quad + 2\frac{(z^3+1)^2}{(z-s)(z-s^{-1})^3} - 2\frac{6z^2(z^3+1)}{(z-s)^2(z-s^{-1})} \\
&\quad \left. -2\frac{6z^2(z^3+1)}{(z-s)(z-s^{-1})^2} + 2\frac{(z^3+1)^2}{(z-s)^2(z-s^{-1})^2}\right]_{z=0} \\
&= s^2 + 1 + s^{-2}
\end{aligned}$$

So all told,

$$I(s) = -\frac{1}{4is} \times 2\pi i \times \left(\frac{(s^3+1)^2}{s^2(s^2-1)} + \frac{s^4+s^2+1}{s^2}\right)$$

$$\boxed{I(s) = \frac{\pi(1+s^3)}{1-s^2}}$$

(iii) Evaluate the integral

$$I = \int_0^\infty \mathrm{d}x \frac{(\ln x)^2}{1+x^2}$$

We have a branch cut in this problem, so we have to be careful with our contours; in particular, to get a closed contour, we have to be sure that we

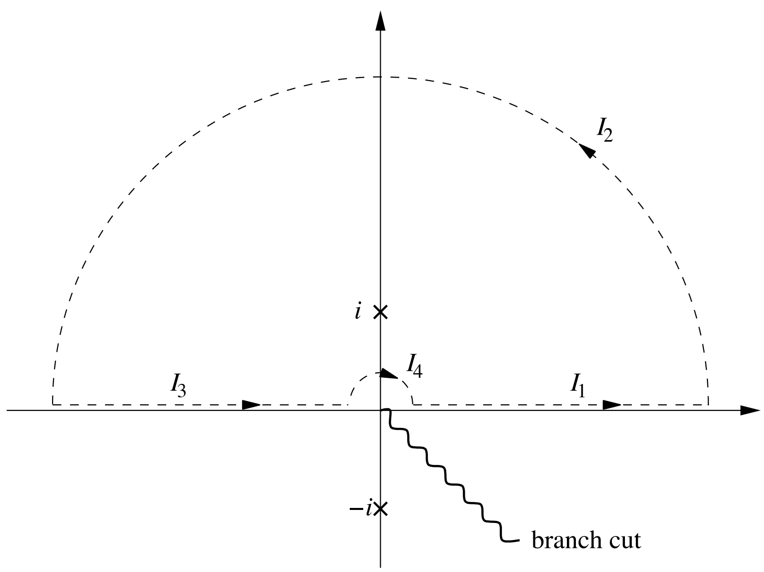

Figure 4: Integration contour and poles for Problem 3(iii).

end on the same Riemann sheet we started on! The structure of the branch cut and singularities is shown in Figure 4; we have drawn the branch cut at an odd angle to get it out of the way. We choose our contour to be the four-part "half-wheel" as shown; denote the radius of the outer half-circle by R and the inner half-circle by ϵ. We will examine each section separately.

Section 1: This is just

$$I_1 = \int_\epsilon^R \mathrm{d}x \frac{(\ln x)^2}{1+x^2}$$

and so

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}} I_1 = I$$

Section 2: On this arc, we have $z = Re^{i\theta}$; plugging this in, we obtain

$$\begin{aligned} I_2 &= \int_0^\pi \mathrm{d}\theta \frac{iRe^{i\theta}(\ln R + i\theta)^2}{1+R^2e^{2i\theta}} \\ &= i\int_0^\pi \mathrm{d}\theta \frac{e^{i\theta}((\ln R)^2 + 2i\theta \ln R - \theta^2)}{R(e^{2i\theta}+R^{-2})} \end{aligned}$$

This is messy, but as $R \to \infty$, we see that this integral will go as $(\ln R)^2/R$ to leading order in R; since $\lim_{R\to\infty}(\ln R)^2/R = 0$, we conclude that

$$\lim_{R\to\infty} I_2 = 0.$$

Section 3: On this path, we have $z = e^{i\pi}x$, and so $\ln z = \ln x + i\pi$. So we have

$$\begin{aligned} I_3 &= \int_R^\epsilon \mathrm{d}x e^{i\pi} \frac{(\ln x + i\pi)^2}{1+x^2} \\ &= \int_\epsilon^R \mathrm{d}x \frac{(\ln x)^2 + 2i\pi \ln x - \pi^2}{1+x^2} \end{aligned}$$

Thus, in the appropriate limit,

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}} I_3 = I + 2\pi i \int_0^\infty \mathrm{d}x \frac{\ln x}{1+x^2} - \pi^2 \int_0^\infty \mathrm{d}x \frac{1}{1+x^2}$$

Section 4: On this path, we have $z = \epsilon e^{i\theta}$, and so we have

$$I_4 = \int_0^\pi \mathrm{d}\theta \frac{i\epsilon e^{i\theta}(\ln\epsilon + i\theta)^2}{1+\epsilon^2 e^{2i\theta}}$$

For $\epsilon \ll 1$, this integral goes as $\epsilon(\ln\epsilon)^2$, which vanishes as $\epsilon \to 0$. Thus,

$$\lim_{\epsilon\to 0} I_4 = 0.$$

All told, we have

$$\oint_C \mathrm{d}z f(z) = 2I + 2\pi i \int_0^\infty \mathrm{d}x \frac{\ln x}{1+x^2} - \pi^2 \int_0^\infty \mathrm{d}x \frac{1}{1+x^2}$$

But the only pole enclosed by this contour is the one at $z = i$; the residue there is

$$\mathrm{Res}(f(z))_{z=i} = \left.\frac{(\ln z)^2}{(z+i)}\right|_{z=i} = -\frac{\pi^2}{8i}$$

since $\ln i = \frac{i\pi}{2}$ on this Riemann sheet. By the Cauchy Residue Theorem, then,

$$2\pi i \times \left(-\frac{\pi^2}{8i}\right) = 2I + 2\pi i \int_0^\infty \mathrm{d}x \frac{\ln x}{1+x^2} - \pi^2 \int_0^\infty \mathrm{d}x \frac{1}{1+x^2}$$

There remains the matter of the two integrals on the right-hand side above. The latter can be evaluated in closed form:

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^2} = \arctan x|_0^\infty = \frac{\pi}{2}$$

This means that we must have

$$2I = \frac{\pi^3}{4} + 2\pi i \int_0^\infty \mathrm{d}x \frac{\ln x}{1+x^2}$$

But since I is real and the integral term on the right-hand side is imaginary (i.e. i times a real integral), we conclude that said last integral term in fact vanishes, and

$$\boxed{I = \frac{\pi^3}{8}}$$

4. (MW A-4) The function $f(z)$ has a pole of order n at $z = z_0$. Show that the function $f'(z)/f(z)$ has a simple pole at z_0. What is the residue?

Since the pole at z_0 is of order n, this implies that the function

$$g(z) = (z - z_0)^n f(z)$$

is regular and non-zero as $z \to z_0$. Since $g(z)$ is regular at z_0, its derivative must be as well:

$$g'(z) = n(z - z_0)^{n-1} f(z) + (z - z_0)^n f'(z)$$

Dividing $g'(z)$ by $g(z)$ then yields

$$\frac{g'(z)}{g(z)} = \frac{n(z - z_0)^{n-1} f(z) + (z - z_0)^n f'(z)}{(z - z_0)^n f(z)}$$

$$\frac{f'(z)}{f(z)} = -\frac{n}{z - z_0} + \frac{g'(z)}{g(z)}$$

Since $g'(z)$ and $g(z)$ are regular at z_0, and $g(z)$ is non-zero, $g'(z)/g(z)$ is regular at z_0; thus, only the first term on the right-hand side above is singular, and $f'(z)/f(z)$ has a simple pole at $z = z_0$ with residue $-n$. □

5. (MW A-5) In this problem, we take $W(x + iy) = U(x, y) + iV(x, y)$ to be an analytic function.

(a) $U(x, y) = e^x \cos y$. What are $V(x, y)$ and $W(z)$?
By the Cauchy-Riemann conditions, we have

$$\frac{\partial V}{\partial y} = \frac{\partial U}{\partial x} = e^x \cos y \qquad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y} = e^x \sin y$$

We conclude that $V(x, y) = e^x \sin y + C$, and

$$W(x + iy) = e^x(\cos y + i \sin y)$$

$$\boxed{W(z) = e^z + iC} \qquad (C \in \mathbb{R})$$

(b) $V(x, y) = y(3x^2 - y^2 - 1)$. What are $U(x, y)$ and $W(z)$?
Same drill: we have

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = 3x^2 - 3y^2 - 1 \qquad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} = -6xy$$

We conclude that $U(x,y) = x^3 - 3xy^2 - x + C$, and

$$W(x+iy) = x^3 + 3ix^2y - 3xy^2 - iy^3 - x - iy + C$$

$$\boxed{W(z) = z^3 - z + C} \qquad (C \in \mathbb{R})$$

(c) $W(z) = \arctan z$. What are $U(x,y)$ and $V(x,y)$?
We first obtain the arctan function in a more useful form. If $W = \arctan z$, then $z = \tan W$, and so

$$\begin{aligned}
\frac{1}{i}\frac{e^{iW} - e^{-iW}}{e^{iW} + e^{-iW}} &= z \\
e^{iW}(1 - iz) &= e^{-iW}(1 + iz) \\
e^{2iW} &= \frac{1+iz}{1-iz} \\
W &= \frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right) + n\pi
\end{aligned}$$

where we've added in the $n\pi$ term because of the usual branch-cut ambiguity.
But

$$\begin{aligned}
1 + iz = ix - y + 1 &= \sqrt{x^2 + (y-1)^2}\exp\left[i\arctan\left(\frac{x}{1-y}\right)\right] \\
1 - iz = -ix + y + 1 &= \sqrt{x^2 + (y+1)^2}\exp\left[-i\arctan\left(\frac{x}{1+y}\right)\right]
\end{aligned}$$

and so

$$\ln\left(\frac{1+iz}{1-iz}\right) = \frac{1}{2}\ln\left(\frac{x^2+(y-1)^2}{x^2+(y+1)^2}\right) + i\left(\arctan\left(\frac{x}{1-y}\right) + \left(\frac{x}{1+y}\right)\right)$$

$$\boxed{U(x,y) = \frac{1}{2}\left(\arctan\left(\frac{x}{1-y}\right) + \arctan\left(\frac{x}{1+y}\right)\right) + n\pi}$$

$$\boxed{V(x,y) = \frac{1}{4}\ln\left(\frac{x^2+(y+1)^2}{x^2+(y-1)^2}\right)}$$

Note that when $y = 0$, $U = \arctan x + n\pi$ and $V = 0$, as would be expected.

6. (MW A-6, a.k.a. the problem of DOOM!) Find the residue of the function $f(z) = z^2 e^{\csc z}$ at the essential singularity $z = \pi$.

If we evaluate the appropriate contour integral numerically, we find that the answer to the problem is approximately -7.576437303. I am unaware of a method to solve this problem analytically, so everyone gets full credit for this problem; if anyone submits a (correct) analytic, closed-form solution to this problem before the end of the quarter, they will get extra credit.

Note: A very common mistake in the attempts to solve this problem was to say something along the lines of

$$\exp\left(\frac{1}{\sin z}\right) = 1 + \frac{1}{\sin z} + \frac{1}{2!}\frac{1}{\sin^2 z} + \frac{1}{3!}\frac{1}{\sin^3 z} + \dots \quad = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots$$

and use this expansion to find the residue. The problem with this is that while $1/\sin^n z$ is indeed z^{-n} to leading order, there will still be sub-leading-order terms that contribute to the residue. For example, we have

$$\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{6} + \dots$$

and so

$$\frac{1}{\sin^3 z} = \frac{1}{z^3} + \frac{1}{2z} + \dots$$

In fact, it's not too hard to see that *every* term in the above expansion of $\exp(1/\sin z)$ will contribute to the residue! This is no small part of why this problem is so hard.