

Problem Set 4: Solutions

Physics 330

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1. (MW 3-10) Evaluate

$$I = \int_0^{\infty} \frac{dx}{1+x^4}.$$

Viewed as a complex function, the integrand has four simple poles, at the roots of the polynomial $z^4 + 1 = 0$; we denote these values by $z_n = e^{(2n-1)i\pi/4}$, where n runs from 1 to 4. We choose the contour shown in Figure 1:

$$\tilde{I} = \oint_C \frac{dz}{1+z^4}$$

This can be decomposed into three parts:

- A line running from 0 to R , along which $z = x$ ($x \in [0, R]$);
- A quarter-circle running from R to iR , along which $z = Re^{i\theta}$ ($\theta \in [0, \frac{\pi}{2}]$);
and
- A line running from iR to 0, along which $z = ix$ ($x \in [R, 0]$).

Splitting this up into three parts, then, we have

$$\begin{aligned} \tilde{I} &= \int_0^R \frac{dx}{1+x^4} + \int_0^{\frac{\pi}{2}} \frac{iRe^{i\theta}d\theta}{1+R^4e^{4i\theta}} + \int_R^0 \frac{idx}{1+(ix)^4} \\ &= (1-i) \int_0^R \frac{dx}{1+x^4} + \int_0^{\frac{\pi}{2}} \frac{iRe^{i\theta}d\theta}{1+R^4e^{4i\theta}} \end{aligned}$$

In the limit where $R \rightarrow \infty$, the second integral will scale as R/R^4 , and so can be neglected:

$$\tilde{I} = (1-i) \int_0^{\infty} \frac{dx}{1+x^4} = (1-i)I$$

Now we simply apply the residue theorem. We have one simple pole enclosed

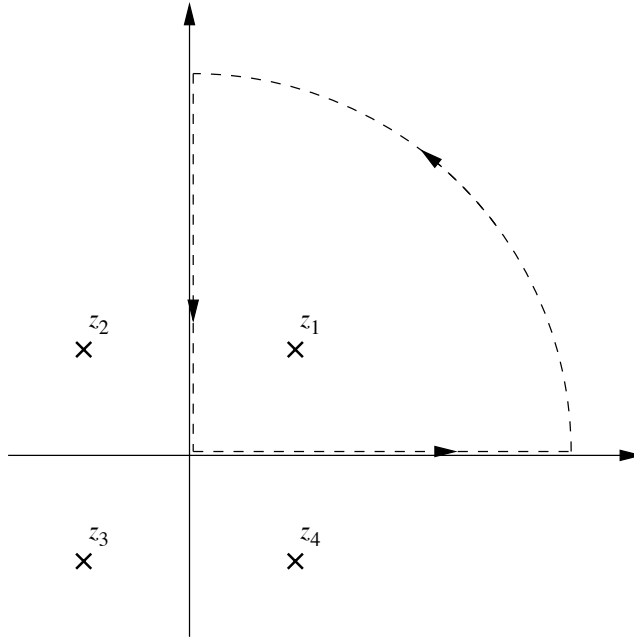


Figure 1: Integration contour for Problem 1.

in the contour, at $z = e^{i\pi/4}$; so the residue is

$$\begin{aligned}
 \operatorname{Res} \left[\frac{1}{1+z^4} \right]_{z=e^{i\pi/4}} &= \left[\frac{z - e^{i\pi/4}}{1+z^4} \right]_{z=e^{i\pi/4}} \\
 &= \frac{1}{(e^{i\pi/4} - e^{-i\pi/4})(e^{i\pi/4} - e^{3i\pi/4})(e^{i\pi/4} - e^{-3i\pi/4})} \\
 &= \frac{1}{(2i)^3 (\sin \frac{\pi}{4})(-e^{i\pi/2} \sin \frac{\pi}{4})(e^{-i\pi/4} \sin \frac{\pi}{2})} \\
 &= -\frac{1+i}{4\sqrt{2}}
 \end{aligned}$$

Thus,

$$\tilde{I} = (1-i)I = 2\pi i \left(-\frac{(1+i)}{4\sqrt{2}} \right)$$

$$I = \frac{\sqrt{2}\pi}{4}$$

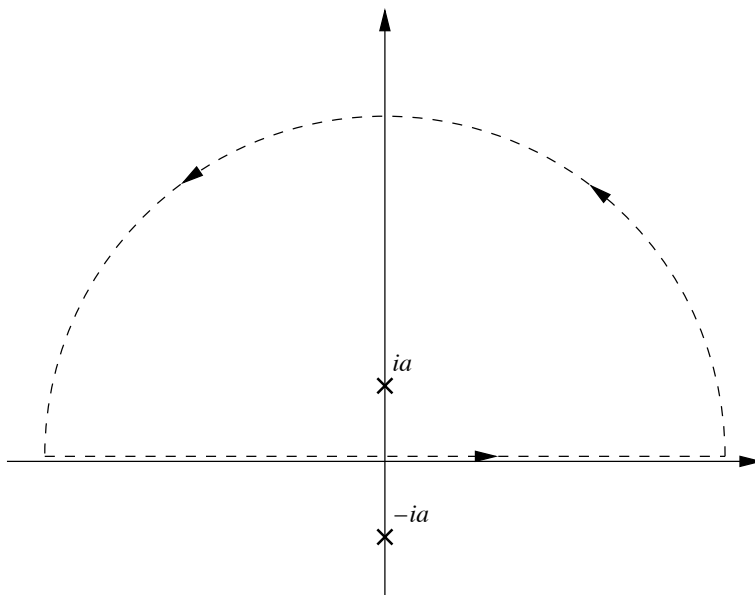


Figure 2: Integration contour for Problem 2.

2. (MW 3-13) Evaluate

$$I = \int \frac{d^3x}{(a^2 + r^2)^3}.$$

Assume that $a > 0$; the $a < 0$ case can be obtained by substituting $a \rightarrow -a$ in the following derivation. In spherical polar coordinates, this is simply

$$\begin{aligned} I &= \iiint \frac{r^2 dr d\theta d\phi}{(a^2 + r^2)^3} \\ &= 4\pi \int_0^\infty \frac{r^2 dr}{(a^2 + r^2)^3} \\ &= 2\pi \int_{-\infty}^\infty \frac{r^2 dr}{(a^2 + r^2)^3} \end{aligned}$$

where we've used the evenness of the integrand in the last step.

We can now use contour integration to evaluate this integral. Viewed as an analytic function, the integrand has poles of order three at $z = \pm ia$. We choose the contour indicated in Figure 2. This consists of two portions, one due to the integration along the real axis and the other due to the upper half-circle:

$$\oint_C \frac{z^2 dz}{(a^2 + z^2)^3} = \int_{-R}^R \frac{r^2 dr}{(a^2 + r^2)^3} + \int_0^\pi \frac{R^2 e^{2i\theta} i R e^{i\theta} d\theta}{(a^2 + R^2 e^{2i\theta})^3}$$

The integral along the upper half-circle will go as R^3/R^6 for $R \gg 1$, and thus in the limit $R \rightarrow \infty$ we have

$$\oint_C \frac{z^2 dz}{(a^2 + z^2)^3} = \int_{-\infty}^{\infty} \frac{r^2 dr}{(a^2 + r^2)^3} = \frac{I}{2\pi}$$

All that remains is to calculate the residues to obtain the contour integral. We have one pole of order three inside the contour, at $z = ia$; the residue there is

$$\begin{aligned} \text{Res} \left[\frac{z^2}{(a^2 + z^2)^3} \right]_{z=ia} &= \frac{1}{2} \frac{d^2}{dz^2} \left[\frac{z^2}{(z + ia)^3} \right]_{z=ia} \\ &= \frac{1}{2} \left[\frac{2}{(z + ia)^3} - \frac{12z}{(z + ia)^4} + \frac{12z^2}{(z + ia)^5} \right]_{z=ia} \\ &= -\frac{i}{16a^3} \end{aligned}$$

Thus,

$$\oint_C \frac{z^2 dz}{(a^2 + z^2)^3} = 2\pi i \left(-\frac{i}{16a^3} \right)$$

$$\boxed{I = \frac{\pi^2}{4a^3}}$$

3. (MW 3-15) Evaluate

$$I = \int_0^{\infty} \frac{x dx}{1 + x^5}.$$

As a complex function, the integrand has five simple poles; we will denote them as $z_n = e^{(2n-1)i\pi/5}$, where n runs from 1 to 5. The symmetry of the denominator of the integrand leads us to choose the contour shown in Figure 3. This contour consists of three segments: the positive real axis from 0 to R , an arc of radius R subtending an angle of $\frac{2\pi}{5}$ radians, and the line parameterized by $z = e^{2\pi i/5}x$, where x runs from R to 0. Splitting this contour integral into these pieces, we have

$$\begin{aligned} \oint_C \frac{z dz}{1 + z^5} &= \int_0^R \frac{x dx}{1 + x^5} + \int_0^{\frac{2\pi}{5}} \frac{iR^2 e^{2i\theta} d\theta}{1 + R^5 e^{5i\theta}} + \int_R^0 \frac{e^{4\pi i/5} x dx}{1 + (e^{2\pi i/5} x)^5} \\ &= (1 - e^{4\pi i/5}) \int_0^R \frac{x dx}{1 + x^5} + \int_0^{\frac{2\pi}{5}} \frac{iR^2 e^{2i\theta} d\theta}{1 + R^5 e^{5i\theta}} \end{aligned}$$

For $R \rightarrow \infty$, the portion of the contour integral due to the arc goes as R^{-3} , and thus in this limit

$$\oint_C \frac{z dz}{1 + z^5} = (1 - e^{4\pi i/5})I$$

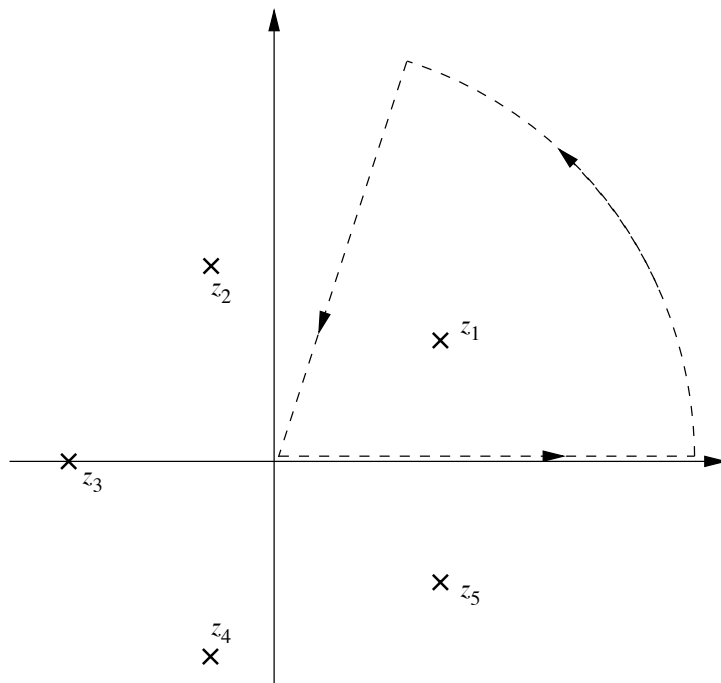


Figure 3: Integration contour for Problem 3.

All that remains is to find the residue. We have one simple pole inside the contour, at $z = z_1 = e^{i\pi/5}$; the residue there is given by

$$\begin{aligned}
 \operatorname{Res} \left[\frac{z}{1+z^5} \right]_{z=e^{i\pi/5}} &= \left[\frac{z(z - e^{i\pi/5})}{1+z^5} \right]_{z=e^{i\pi/5}} \\
 &= \frac{e^{i\pi/5}}{(e^{i\pi/5} - e^{3i\pi/5})(e^{i\pi/5} - e^{i\pi}) (e^{i\pi/5} - e^{7i\pi/5})(e^{i\pi/5} - e^{9i\pi/5})} \\
 &= \frac{e^{i\pi/5}}{(-2i)^4 (e^{2\pi i/5} \sin \frac{\pi}{5})(e^{3\pi i/5} \sin \frac{2\pi}{5})(e^{4\pi i/5} \sin \frac{3\pi}{5})(e^{5\pi i/5} \sin \frac{4\pi}{5})} \\
 &= -\frac{e^{2\pi i/5}}{16 \sin \frac{\pi}{5} \sin \frac{2\pi}{5} \sin \frac{3\pi}{5} \sin \frac{4\pi}{5}}
 \end{aligned}$$

We can then use the curious trigonometric identity¹

$$\prod_{m=1}^{n-1} \sin \frac{m\pi}{n} = \frac{n}{2^{n-1}}$$

¹I've found this result cited in several places; I'd love to cite a proof for it, but I can't find one easily and haven't been able to prove it for general n .

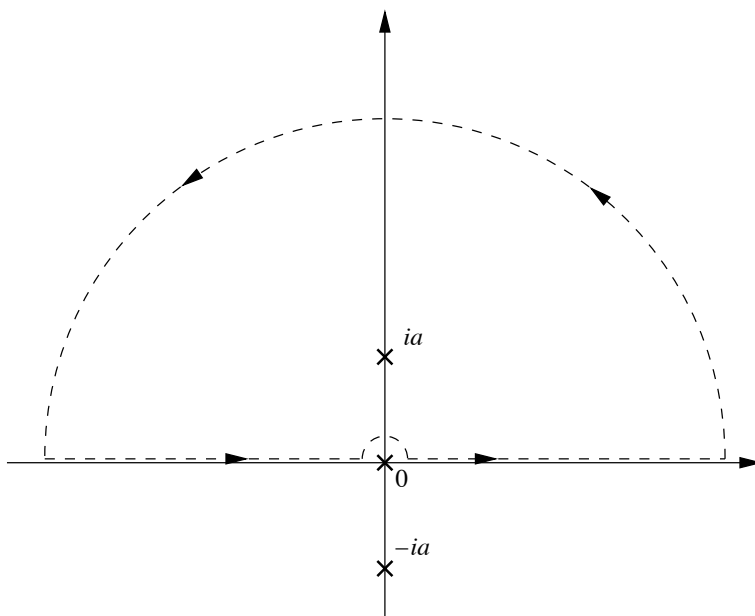


Figure 4: Integration contour for Problem 4.

with $n = 5$ to obtain

$$\text{Res} \left[\frac{z}{1+z^5} \right]_{z=e^{i\pi/5}} = -\frac{e^{2\pi i/5}}{5}$$

So we have

$$\oint_C \frac{z dz}{1+z^5} = (1 - e^{4\pi i/5})I = 2\pi i \left(-\frac{e^{2\pi i/5}}{5} \right)$$

$$I = \frac{2ie^{2\pi i/5}}{e^{4\pi i/5} - 1} \frac{\pi}{5}$$

$$\boxed{I = \frac{\pi}{5 \sin \frac{2\pi}{5}}}$$

4. (MW 3-22) Evaluate

$$I = \int_0^\infty \frac{\sin x dx}{x(a^2 + x^2)}$$

Assume that $a > 0$; the $a < 0$ case can be obtained by substituting $a \rightarrow -a$ in the following derivation. Consider the related integral

$$I' = \oint_C \frac{e^{iz} dz}{z(a^2 + z^2)}$$

where C is the contour shown in Figure 4, consisting of four segments:

- the real axis from ϵ to R ;
- a half-circle with $|z| = R$;
- the real axis from $-R$ to $-\epsilon$; and
- a half-circle with $|z| = \epsilon$.

(The reason for this choice of contour will become evident below.) As usual, we split this integral into its component parts:

$$\begin{aligned}
I' &= \int_{\epsilon}^R \frac{e^{ix} dx}{x(a^2 + x^2)} + \int_0^{\pi} \frac{\exp(iRe^{i\theta})iRe^{i\theta} d\theta}{Re^{i\theta}(a^2 + R^2e^{2i\theta})} + \int_{-R}^{-\epsilon} \frac{e^{ix} dx}{x(a^2 + x^2)} \\
&\quad + \int_{\pi}^0 \frac{\exp(i\epsilon e^{i\theta})i\epsilon e^{i\theta} d\theta}{\epsilon e^{i\theta}(a^2 + \epsilon^2 e^{2i\theta})} \\
&= \int_{\epsilon}^R \frac{e^{ix} dx}{x(a^2 + x^2)} + \int_R^{\epsilon} \frac{e^{-i\tilde{x}} d\tilde{x}}{\tilde{x}(a^2 + \tilde{x}^2)} + i \int_0^{\pi} \frac{\exp(iRe^{i\theta}) d\theta}{(a^2 + R^2e^{2i\theta})} + i \int_{\pi}^0 \frac{\exp(i\epsilon e^{i\theta}) d\theta}{(a^2 + \epsilon^2 e^{2i\theta})} \\
&= \int_{\epsilon}^R \frac{(e^{ix} - e^{-ix}) dx}{x(a^2 + x^2)} + i \int_0^{\pi} \frac{\exp(iRe^{i\theta}) d\theta}{(a^2 + R^2e^{2i\theta})} + i \int_{\pi}^0 \frac{\exp(i\epsilon e^{i\theta}) d\theta}{(a^2 + \epsilon^2 e^{2i\theta})}
\end{aligned}$$

(where we have substituted $\tilde{x} = -x$ in the second step.) In the limit $\epsilon \rightarrow 0$, $R \rightarrow \infty$, the first integral becomes $2i$ times our desired integral I . For the second integral, we note that $\Im(e^{i\theta}) > 0$ in the upper half-plane; thus, $\Re(iRe^{i\theta})$ goes to $-\infty$ as $R \rightarrow \infty$, and so the exponential in the integral will vanish in this limit. (This is why we chose to close the integral in the upper half-plane rather than the lower half-plane: in the lower half-plane, this contribution would diverge instead.) Finally, the third integral has a well-defined limit as $\epsilon \rightarrow 0$, namely

$$\lim_{\epsilon \rightarrow 0} \int_{\pi}^0 \frac{\exp(i\epsilon e^{i\theta}) d\theta}{(a^2 + \epsilon^2 e^{2i\theta})} = \int_{\pi}^0 \frac{d\theta}{a^2} = -\frac{\pi}{a^2}.$$

All told, we have

$$I' = 2iI - \frac{i\pi}{a^2}$$

All that's left is to find the residues. In this case, we have one simple pole at $z = ia$; the residue there is simply

$$\text{Res} \left[\frac{e^{iz}}{z(a^2 + z^2)} \right]_{z=ia} = \left[\frac{e^{iz}}{z(z + ia)} \right]_{z=ia} = -\frac{e^{-a}}{2a^2}$$

and so we have

$$I' = 2iI - \frac{i\pi}{a^2} = -i\pi \frac{e^{-a}}{a^2}$$

$$\boxed{I = \frac{\pi}{2a^2}(1 - e^{-a})}$$

Note: It is important to note here that changing from $\sin z$ to e^{iz} is essential to this method of solving the problem: for large imaginary z , we have $\sin(i|z|) = i \sinh(|z|) \approx ie^{|z|}$ as $|z| \rightarrow \infty$, and this exponential growth will dominate the decay caused by any polynomials we might have in the denominator. We can still keep the $\sin z$ around, but this requires splitting it into an integral involving e^{iz} and one involving e^{-iz} ; the former contour is then closed in the upper complex half-plane, while the latter would be closed in the lower half-plane.

5. “Clausen’s Paradox.”

Complex exponentiation is a tricky business, mainly because it involves logarithms. From our experience with real numbers, we usually write

$$w^z = \exp[z \ln w] = \exp[z(\ln |w| + i(\arg(w) + 2\pi m))]$$

where $-\pi \leq \arg(w) < \pi$ (defining our branch cut to lie on the negative real axis) and we have defined the function $\exp(z)$ to be the infinite series

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

Note that we have a free choice in our definition of w^z : we can freely specify which branch of the logarithm we’re using by varying the integer m . However, different values of m can lead to different answers for w^z ; for example, we have

$$1^i = \exp[i(0 + 2\pi m)] = e^{-2\pi m}$$

Any one of these values corresponding to different choices of m can be viewed as “correct”; but once we’ve chosen a branch that we’re working on, we have to be careful that we don’t accidentally slip on to another branch or equate values on two different branches.

With this in mind, let’s resolve the paradox. The problem with this “proof” is between the third and fourth step: it is implicitly assumed that

$$(e^{z_1})^{z_2} = e^{z_1 z_2}.$$

From the definition of complex exponentiation above, we have

$$(e^{z_1})^{z_2} = \exp[z_2(\ln |e^{z_1}| + i(\arg(e^{z_1}) + 2\pi m))]$$

We know that $\ln |e^{z_1}| = \Re(z_1)$; however, $\arg(e^{z_1}) \neq \Im(z_1)$, since $\arg(z)$ is defined to lie between $-\pi$ and π . Instead, we have

$$\arg(e^{z_1}) = \Im(z_1) - 2\pi p$$

where p is the unique integer which makes the right-hand side lie between $-\pi$ and π ; in other words,

$$p = \left\lfloor \frac{\Im(z_1) + \pi}{2\pi} \right\rfloor.$$

All told, we have

$$(e^{z_1})^{z_2} = \exp[z_2(z_1 + 2\pi i(m - p))]$$

In particular, if $z_1 = z_2 = 1 + 2\pi in$, then $p = n$ and we have

$$\begin{aligned} (e^{1+2\pi n})^{1+2\pi n} &= \exp[(1 + 2\pi in)^2 + 2\pi i(1 + 2\pi in)(m - n)] \\ &= \exp[1 + 2\pi in + 2\pi im(1 + 2\pi in)] \end{aligned}$$

We could, of course, be ornery and decide to define exponentiation with $m = n$, and the “paradox” would then seem to carry through as before. But if we did this, then the very first step would be false: we would have

$$\begin{aligned} e^{2\pi in} &= \exp[2\pi in(\ln |e| + i(\arg(e) + 2\pi m))] \\ &= \exp[2\pi in(1 + 2\pi im)] \\ &= \exp[2\pi in - 4\pi^2 mn] \neq 1 \end{aligned}$$

and the whole logical chain can’t get off the ground unless $m = 0$. (In other words, it’s rather silly to prove a “paradox” by assuming that, say, $1 = 2$ in the first step.) We conclude that

$$(e^{1+2\pi n})^{1+2\pi n} = \exp[1 + 2\pi in] = e$$

and complex mathematics is saved.

Note: It might be thought that the logical inference in the third step is also incorrect: the proof seems to rely on the fact that $x = y \Rightarrow x^z = y^z$. It is true that this definition is ambiguous — we always have the choice of m due to the logarithm — but once m is chosen, the equation is unambiguous and true. What’s more, there is a natural choice of m for $x, y \in \mathbb{R}$, which is the case in question: it is almost always implicitly defined with $m = 0$, as noted above. See §1.4 of Carrier, Krook, and Pearson for further discussion.

6. (MW 3-31)

- (a) Consider the contour integral $\oint f(z) \cot \pi z \, dz$ around a suitable large contour, and obtain thereby a formula for the sum

$$\sum_{n=-\infty}^{\infty} f(n).$$

To do this problem, we have to assume a couple of things: first, that $f(n)$ is well-defined for all n , i.e., $f(z)$ doesn’t have a poles at any integer value; and second, that $f(z) \rightarrow 0$ faster than $|z|^{-1}$ as $|z| \rightarrow \infty$.

With these assumptions, let a_1, a_2, \dots, a_m be the poles of $f(z)$, and let b_1, b_2, \dots, b_m be their respective residues. The function $f(z) \cot \pi z$ will

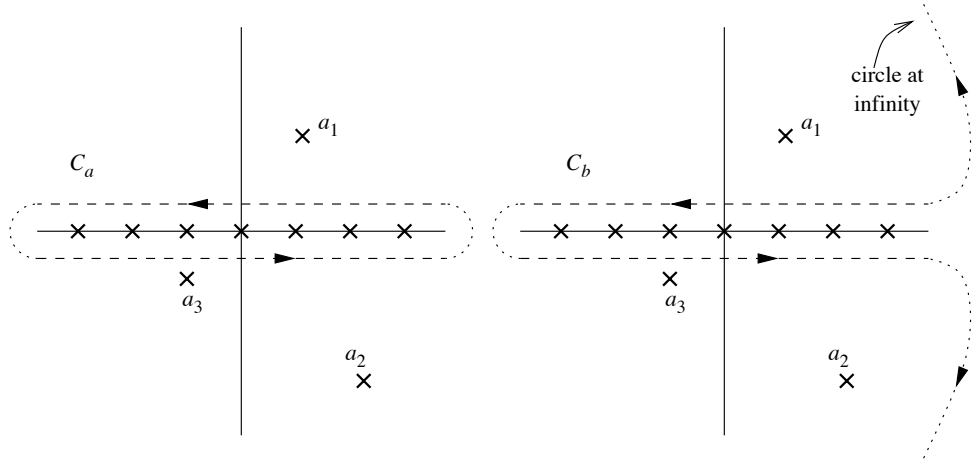


Figure 5: Integration contours for Problem 6.

then have two types of poles: those due to the poles of $f(z)$, at which the residues will be

$$\operatorname{Res}[f(z) \cot \pi z]_{z=a_i} = \cot(\pi a_i) \operatorname{Res}[f(z)]_{z=a_i} = b_i \cot \pi a_i,$$

and those due to the poles of $\cot \pi z$, at which the residues will be

$$\operatorname{Res}[f(z) \cot \pi z]_{z=n} = f(n) \operatorname{Res}[\cot \pi z]_{z=n} = \frac{1}{\pi} f(n),$$

Now consider the integrals around the contours shown in Figure 5. The left-hand contour C_a encloses all the poles located at $z \in \mathbb{Z}$; thus, the integral about the left-hand contour will be equal to $2\pi i$ times the sum of the residues at the integers:

$$\oint_{C_a} f(z) \cot \pi z \, dz = 2\pi i \times \frac{1}{\pi} \sum_{n=-\infty}^{\infty} f(n)$$

Meanwhile, the right-hand contour C_b encloses all the poles *not* located at the integers, and circles them in the clockwise direction; thus, we will have

$$\oint_{C_b} f(z) \cot \pi z \, dz = -2\pi i \times \sum_{i=1}^m b_i \cot \pi a_i$$

But since $f(z)$ dies off as $|z| \rightarrow \infty$ (or alternately, since $f(z)$ does not have a pole at ∞), we can deform C_a into C_b via the addition of a “circle at infinity” (cf. Matthews & Walker, pp. 74–76), and thus these contour integrals are equal:

$$\oint_{C_a} f(z) \cot \pi z \, dz = \oint_{C_b} f(z) \cot \pi z \, dz$$

$$\boxed{\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{i=1}^m b_i \cot \pi a_i}$$

(b) Evaluate

$$g(a) = \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2}$$

Assume that $a > 0$; the $a < 0$ case can be obtained by substituting $a \rightarrow -a$ in the following derivation. From the above problem, all we need to do is to find the residues and locations of the function

$$f(z) = \frac{1}{z^2 + a^2}$$

The poles of this function are located at $z \pm ia$, and the residues there are

$$\text{Res}[f(z)]_{z=ia} = -\frac{i}{2a}$$

and

$$\text{Res}[f(z)]_{z=-ia} = \frac{i}{2a}$$

We can now apply the above formula:

$$\begin{aligned} g(a) &= \sum_{n=-\infty}^{\infty} f(n) \\ &= -\pi \left[-\frac{i}{2a} \cot(i\pi a) + \frac{i}{2a} \cot(-i\pi a) \right] \end{aligned}$$

$$\boxed{g(a) = \frac{\pi}{a} \coth \pi a}$$

Note: Although I have drawn the contours C_a and C_b in Figure 5 running parallel to the real axis, they need not necessarily do so; all that is necessarily is that all the poles of $f(z)$ be on one side of the line, and all the poles of $\cot \pi z$ be on the other. In particular, $f(z)$ can have poles at non-integer real values for this method to work; the upper part of the contour can simply “dip down” below the real axis so that these poles are above the contour.