Problem Set 5: Solutions

Physics 330

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1. Repeat the derivation of the Cauchy Principal Value formula using a contour closed in the lower half-plane. Show that this formula is equivalent to the one derived in class.

For the sake of argument, we will assume that we have only one simple pole on the real axis (as was assumed in class.) Let C be the contour pictured in Figure 1. We can split this contour integral up into three parts: C_1, the portions of the contour lying on the real axis; C_2, a half-circle of radius ϵ centered on the pole at z_0; and C_3, a half-circle of radius R lying in the lower half-plane. In other words,

$$\oint_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z + \int_{C_3} f(z)\,\mathrm{d}z.$$

Now,

$$\int_{C_1} f(z)\,\mathrm{d}z = \int_{-R}^{z_0-\epsilon} f(x)\,\mathrm{d}x + \int_{z_0+\epsilon}^{R} f(x)\,\mathrm{d}x$$

and so the integral along C_1 will go to the desired quantity in the limit $\epsilon \to 0, R \to \infty$. From the derivation in class (or from Mathews & Walker), we also know that in the limit $\epsilon \to 0$,

$$\int_{C_2} f(z)\,\mathrm{d}z = -i\pi \mathrm{Res}[f(z)]_{z=z_0}.$$

Finally, for the last integral, we have $z = Re^{i\theta}$, and so

$$\int_{C_3} f(z)\,\mathrm{d}z = \int_0^{-\pi} f(z) iRe^{i\theta}\,\mathrm{d}\theta$$

But

$$\left| \int_0^{-\pi} f(z) iRe^{i\theta}\,\mathrm{d}\theta \right| \leq \pi R \max_{\theta\in[-\pi,0]} |f(z)|$$

and since $R|f(z)| \to 0$ as $R \to \infty$, the contribution from C_3 will vanish in this limit, leaving

$$\oint_C f(z)\,\mathrm{d}z = \mathrm{P}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - i\pi \mathrm{Res}[f(z)]_{z=z_0}.$$

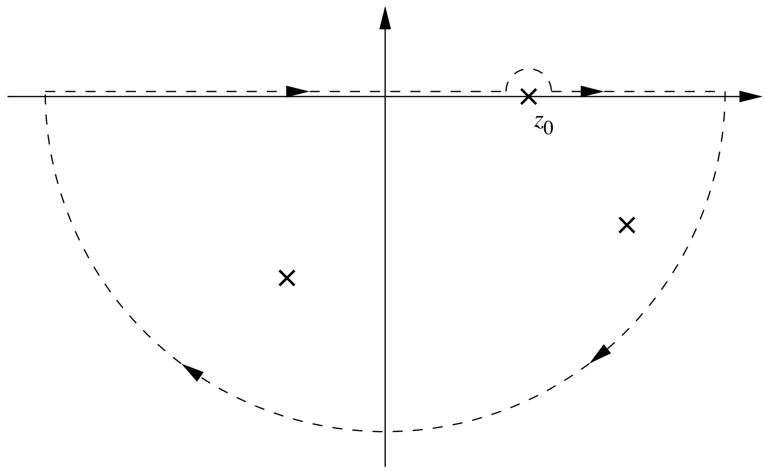

Figure 1: Integration contour for Problem 1.

We can now apply the residue theorem to find the desired quantity. The contour C encloses all the poles in the lower half-plane, as well as z_0; since it circles them in a clockwise direction, we have

$$-2\pi i \left\{ \mathrm{Res}[f(z)]_{z=z_0} + \sum_{\Im(z_i)<0} \mathrm{Res}[f(z)]_{z=z_i} \right\} = \mathrm{P}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - i\pi \mathrm{Res}[f(z)]_{z=z_0}$$

where the summation on the right-hand side runs over all poles in the lower half-plane. Rearranging yields our final answer:

$$\boxed{\mathrm{P}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = -2\pi i \left(\frac{1}{2}\mathrm{Res}[f(z)]_{z=z_0} + \sum_{\Im(z_i)<0} \mathrm{Res}[f(z)]_{z=z_i} \right)}$$

In class (and in Mathews & Walker), we had instead

$$\mathrm{P}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \left(\frac{1}{2}\mathrm{Res}[f(z)]_{z=z_0} + \sum_{\Im(z_i)>0} \mathrm{Res}[f(z)]_{z=z_i} \right)$$

Superficially, these formulas would seem to be different. However, suppose we were to perform a contour integral of f around a complete circle C' of radius R:

$$\oint_{C'} f(z)\,\mathrm{d}z = \int_0^{2\pi} f(z) iRe^{i\theta}\,\mathrm{d}\theta$$

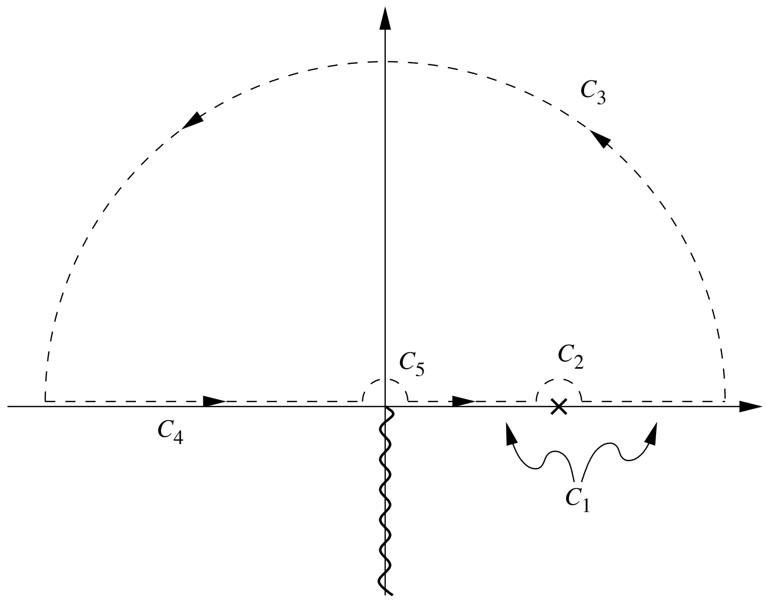

Figure 2: Integration contour for Problem 2.

By similar logic to that used for C_3 above, this entire integral will vanish as $R \to \infty$; by the residue theorem, then, we have

$$\sum_i \operatorname{Res}[f(z)]_{z=z_i} = 0$$

or, rearranging the terms and multiplying both sides by $2\pi i$,

$$2\pi i \left(\frac{1}{2} \operatorname{Res}[f(z)]_{z=z_0} + \sum_{\Im(z_i)>0} \operatorname{Res}[f(z)]_{z=z_i} \right) \\ = -2\pi i \left(\frac{1}{2} \operatorname{Res}[f(z)]_{z=z_0} + \sum_{\Im(z_i)<0} \operatorname{Res}[f(z)]_{z=z_i} \right)$$

Thus, the two formulas are equivalent after all (as long as $f(z)$ falls off fast enough.) □

2. Evaluate

$$\mathrm{P} \int_0^\infty \frac{x^{s-1}}{1-x} \, \mathrm{d}x$$

where $0 < s < 1$.

For the non-integer values of s we are concerned with, $f(z) = z^{s-1}/(1-z)$ has a pole at $z = 1$ and a branch cut running from 0 to ∞; we take this branch cut to lie on the negative imaginary axis. Further, $f(z)$ is not well-defined at $z = 0$, so we must integrate around this point.

With these constraints in mind, we use the contour C depicted in Figure 2. We can split this up into five parts, as indicated in the figure:

$\boldsymbol{C}_1$: This is just the two segments of the C on the positive real axis:

$$\int_{C_1} \frac{z^{s-1}}{1-z}\,\mathrm{d}z = \int_{\epsilon_0}^{1-\epsilon_1} \frac{x^{s-1}}{1-x}\,\mathrm{d}x + \int_{1+\epsilon_1}^{R} \frac{x^{s-1}}{1-x}\,\mathrm{d}x$$

where ϵ_0 is the radius of the half-circle comprising C_5 and ϵ_1 is the radius of the half-circle comprising C_2. In the limit $\epsilon_0, \epsilon_1 \to 0$, $R \to \infty$, of course,

$$\lim_{\substack{\epsilon_0,\epsilon_1\to 0\\ R\to\infty}} \int_{C_1} \frac{z^{s-1}}{1-z}\,\mathrm{d}z = \mathrm{P}\int_0^\infty \frac{x^{s-1}}{1-x}\,\mathrm{d}x.$$

$\boldsymbol{C}_2$: This is a half-circle of radius ϵ_1 running above the simple pole at $z = 1$. For such a path, we have (see p. 481 of Mathews & Walker)

$$\int_{C_2} \frac{z^{s-1}}{1-z}\,\mathrm{d}z = -i\pi \mathrm{Res}\left[\frac{z^{s-1}}{1-z}\right]_{z=1} = -i\pi[-z^{s-1}]_{z=1} = i\pi$$

in the limit $\epsilon_1 \to 0$.

$\boldsymbol{C}_3$: This is a half-circle of radius R centered at the origin. Along this contour, we have $z = Re^{i\theta}$, and so

$$\begin{aligned}\int_{C_3} \frac{z^{s-1}}{1-z}\,\mathrm{d}z &= \int_0^\pi \frac{R^{s-1}e^{i\theta(s-1)}}{1-Re^{i\theta}} iRe^{i\theta}\,\mathrm{d}\theta \\ &= iR^{s-1}\int_0^\pi \frac{e^{is\theta}}{\frac{1}{R}-e^{i\theta}}\,\mathrm{d}\theta\end{aligned}$$

The integrand above is bounded for $R \neq 1$, and so in the limit $R \to \infty$,

$$\lim_{R\to\infty} \int_{C_3} \frac{z^{s-1}}{1-z}\,\mathrm{d}z = 0.$$

$\boldsymbol{C}_4$: This portion of the contour consists of the negative real axis, along which $z = xe^{i\pi}$. Thus,

$$\begin{aligned}\int_{C_4} \frac{z^{s-1}}{1-z}\,\mathrm{d}z &= \int_R^{\epsilon_0} \frac{x^{s-1}e^{i(s-1)\pi}}{1-xe^{i\pi}} e^{i\pi}\,\mathrm{d}x \\ &= -e^{is\pi}\int_{\epsilon_0}^R \frac{x^{s-1}}{1+x}\,\mathrm{d}x,\end{aligned}$$

and in the appropriate limit,

$$\lim_{\substack{\epsilon_0 \to 0 \\ R \to \infty}} \int_{C_4} \frac{z^{s-1}}{1-z}\,\mathrm{d}z = -e^{is\pi} \int_0^\infty \frac{x^{s-1}}{1+x}.$$

$\boldsymbol{C_5}$: This is a half-circle with radius ϵ_0 above the origin; along this path, we have

$$\begin{aligned}\int_{C_5} \frac{z^{s-1}}{1-z}\,\mathrm{d}z &= \int_\pi^0 \frac{\epsilon_0^{s-1} e^{i\theta(s-1)}}{1-\epsilon_0 e^{i\theta}} i\epsilon_0 e^{i\theta}\,\mathrm{d}\theta \\ &= i\epsilon_0^s \int_\pi^0 \frac{e^{is\theta}}{1-\epsilon_0 e^{i\theta}}\,\mathrm{d}\theta\end{aligned}$$

The integrand is bounded as long as $\epsilon_0 \neq 1$, so in the limit $\epsilon_0 \to 0$,

$$\lim_{\epsilon_0 \to 0} \int_{C_5} \frac{z^{s-1}}{1-z}\,\mathrm{d}z = 0.$$

However, the contour C doesn't enclose any poles; so adding the pieces together and taking the desired limits, we have

$$\mathrm{P}\int_0^\infty \frac{x^{s-1}}{1-x}\,\mathrm{d}x + i\pi - (\cos s\pi + i\sin s\pi)\int_0^\infty \frac{x^{s-1}}{1+x}\,\mathrm{d}x = 0$$

To extract the desired integral (i.e. the first term) from this equation, we simply equate real and imaginary parts. Since both integrals are real, the imaginary part of the above equation is

$$i\pi - i\sin s\pi \int_0^\infty \frac{x^{s-1}}{1+x}\,\mathrm{d}x = 0 \quad \Rightarrow \quad \int_0^\infty \frac{x^{s-1}}{1+x}\,\mathrm{d}x = \frac{\pi}{\sin s\pi}$$

and the real part of the equation then gives us our result:

$$\mathrm{P}\int_0^\infty \frac{x^{s-1}}{1-x}\,\mathrm{d}x - \cos s\pi \int_0^\infty \frac{x^{s-1}}{1+x} = 0$$

$$\boxed{\mathrm{P}\int_0^\infty \frac{x^{s-1}}{1-x}\,\mathrm{d}x = \pi\cot s\pi}$$

3. Let $f(z)$ be a function analytic inside a contour C except for a finite number of poles.

(i) Evaluate

$$I = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

in terms of the number of poles and zeroes of $f(z)$.

The only ways that $f'(z)/f(z)$ can have a pole is if $f(z)$ has a pole or if $f(z)$ has a zero. We previously did the pole case: from Problem 4 on Problem Set 3 (MW A-4), we know that if $f(z)$ has a pole of order n at z_0, then

$$\text{Res}\left[\frac{f'(z)}{f(z)}\right]_{z=z_0} = -n.$$

We can essentially duplicate the proof of this fact for the zeroes of $f(z)$. Let y_0 be such that $f(y_0) = 0$. Since $f(z)$ is analytic and vanishes at y_0, its Laurent expansion about $z = y_0$ will start at some non-zero term $a_m(z - y_0)^m$, where m is the order of the zero. This implies that the function

$$g(z) = \frac{f(z)}{(z - y_0)^m}$$

is regular and non-zero as $z \to y_0$. Since $g(z)$ is regular at y_0, its derivative must be as well:

$$g'(z) = -m\frac{f(z)}{(z - y_0)^{m+1}} + \frac{f'(z)}{(z - y_0)^m}$$

Dividing $g'(z)$ by $g(z)$ then yields

$$\frac{g'(z)}{g(z)} = -\frac{m}{z - y_0} + \frac{f'(z)}{f(z)}$$

$$\frac{f'(z)}{f(z)} = \frac{m}{z - y_0} + \frac{g'(z)}{g(z)}$$

Since $g'(z)$ and $g(z)$ are regular at y_0, and $g(z)$ is non-zero, $g'(z)/g(z)$ is regular at y_0; thus, only the first term on the right-hand side above is singular, and $f'(z)/f(z)$ has a simple pole at $z = y_0$ with residue m.

All told, every zero contributes its order to the desired integral, and every pole subtracts its order from the desired integral:

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{y_i}\begin{pmatrix}\text{order of the}\\ \text{zero at } y_i\end{pmatrix} - \sum_{z_i}\begin{pmatrix}\text{order of the}\\ \text{pole at } z_i\end{pmatrix}$$

where the y_i's are the zeroes of $f(z)$ in C and the z_i's are its poles. This can be simplified if we view a zero of order m as counting for m zeroes, and a pole of order n as counting for n poles:

$$\boxed{\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = \begin{pmatrix}\text{number of}\\ \text{zeroes in } C\end{pmatrix} - \begin{pmatrix}\text{number of}\\ \text{poles in } C\end{pmatrix}}$$

(ii) Prove that a polynomial of degree n has n roots in $\mathbb{C}$.

Let $f(z)$ be a polynomial of degree n:

$$f(z) = \sum_{m=0}^{n} a_m z^m$$

Then

$$f'(z) = \sum_{m=0}^{n} m a_m z^{m-1}$$

Consider now the integral of $f'(z)/f(z)$ about a circle C of radius R centered at the origin:

$$\begin{aligned}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z &= \int_0^{2\pi} \frac{\sum_{m=0}^{n} m a_m (Re^{i\theta})^{m-1}}{\sum_{m=0}^{n} a_m (Re^{i\theta})^m} iRe^{i\theta}\,\mathrm{d}\theta \\ &= i\int_0^{2\pi} \frac{\sum_{m=0}^{n} m a_m R^{m-n} e^{im\theta}}{\sum_{m=0}^{n} a_m R^{m-n} e^{im\theta}}\,\mathrm{d}\theta\end{aligned}$$

where we have divided numerator and denominator by R^n in the second step. Since $m \leq n$ in each sum, $R^{m-n} \to 0$ as $R \to 0$ unless $m = n$, and only the highest-order term in each sum will survive in this limit:

$$\lim_{R\to\infty} \oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = i\int_0^{2\pi} \frac{n a_n e^{in\theta}}{a_n e^{in\theta}}\mathrm{d}\theta = i\int_0^{2\pi} n\,\mathrm{d}\theta = 2\pi i n$$

Thus, from the previous problem, we conclude that

$$\begin{pmatrix}\text{number of} \\ \text{zeroes of } f(z)\end{pmatrix} - \begin{pmatrix}\text{number of} \\ \text{poles of } f(z)\end{pmatrix} = n$$

But $f(z)$ is analytic everywhere; so it has no poles, and we conclude that $f(z)$ must have n zeroes in the complex plane. □

4. Let $f(z)$ be an analytic function inside some closed contour. Show that $|f(z)|$ attains its maximum value on the contour itself.

We will construct a proof by contradiction. Denote the contour in question by C and its interior by Σ. Suppose that $|f(z)|$ doesn't attain its maximum value on the contour C; in other words, for some $z_0 \in \Sigma$, $|f(z_0)| \geq |f(z)|$ for all $z \in \Sigma$, and $|f(z_0)| > |f(z)|$ for all z lying on C.

We know that $f(z_0)$ is given by the formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{C'} \frac{f(z)}{z - z_0}$$

where C' is any contour enclosing z_0. In particular, if C' is a circle of radius r

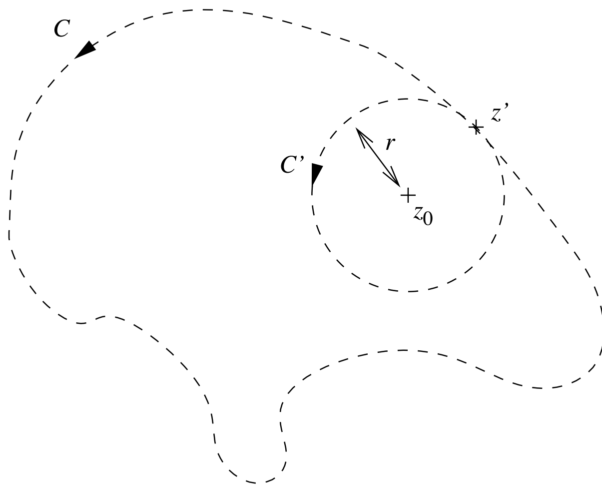

Figure 3: Integration contours for Problem 4.

centered on z_0 (i.e. $z = z_0 + re^{i\theta}$), we must have

$$\begin{aligned}|f(z_0)| &= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}} ire^{i\theta}\,\mathrm{d}\theta\right| \\ &= \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0+re^{i\theta})\,\mathrm{d}\theta\right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\,\mathrm{d}\theta\end{aligned}$$

But if we pick r to be the shortest distance between C and z_0 (see Figure 3), $C' \subset C \cup \Sigma$; in this case, $|f(z_0+re^{i\theta})| \le |f(z_0)|$, and we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\,\mathrm{d}\theta \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,\mathrm{d}\theta = |f(z_0)|$$

with equality only holding here if $|f(z_0+re^{i\theta})| = |f(z_0)|$ for all θ. Moreover, this equality *must* be satisfied; otherwise, our two inequalities above would imply that $|f(z_0)| < |f(z_0)|$. We conclude that

$$|f(z_0+re^{i\theta})| = |f(z_0)|$$

for all points on C'. C' intersects C in at least one point; call this point z'. Since $z' \in C'$, $|f(z')| = |f(z_0)|$. But we assumed that $|f(z)|$ did *not* attain its maximum value on C, and since $z' \in C$ as well, $|f(z')| < |f(z_0)|$. $\rightarrow\!\leftarrow$

We conclude that if $f(z)$ is analytic inside some contour, it must attain its maximum on that contour. $\square$

Note: In fact, it is possible to show that any entire function for which $|f(z)|$ has a local maximum must actually be *constant*. A sketch of the proof is as follows: If $|f(z)|$ is a local maximum, then there must be some $\epsilon > 0$ such that $|z - z_0| < \epsilon \Rightarrow |f(z)| \leq |f(z_0)|$. By the above logic, the value of $|f(z)|$ on any circular contour with radius δ must be constant and equal to $|f(z_0)|$. It can be shown using the Cauchy-Riemann conditions that if $f(z)$ is analytic and $|f(z)|$ is constant, then $f(z)$ itself must also be constant. Thus, $f(z) = f(z_0)$ in an open ball of radius ϵ surrounding z_0. We can then analytically continue the function from this ball to the entire plane, and it will be constant everywhere.

5. Show that there exists a solution of Bessel's equation ($m = 0$) of the form

$$J_0(x)\ln x + Ax^2 + Bx^4 + Cx^6 + \ldots$$

and determine A, B, and C.

We solve this in a similar method to that used for $m > 0$, $m \in \mathbb{Z}$ in Mathews & Walker. For $m = 0$, we have

$$y(x,s) = x^s\left[1 - \frac{x^2}{(s+2)^2} + \frac{x^4}{(s+2)^2(s+4)^2} - \frac{x^6}{(s+2)^2(s+4)^2(s+6)^2} + \ldots\right]$$

and so denoting $\mathcal{L}$ to be the Bessel differential operator (with $m = 0$), we have

$$\mathcal{L}y(x,s) = s^2x^s.$$

This means that $y(x,0)$ is a solution of the Bessel's equation, and in fact, $y(x,0) = J_0(x)$. Further, we have

$$\mathcal{L}\left[\frac{\partial}{\partial s}y(x,s)\right] = \frac{\partial}{\partial s}\mathcal{L}y(x,s) = 2sx^s + s^2x^s\ln x$$

which also vanishes when $s = 0$. (Note that this wouldn't have worked for $m \neq 0$ — in that case, the s^2 would have become an $s^2 - m^2$.) Thus,

$$\left.\frac{\partial}{\partial s}y(x,s)\right|_{s=0}$$

is another solution of Bessel's equation with $m = 0$.

Computing this solution, we have

$$\begin{aligned}\frac{\partial}{\partial s}y(x,s) &= x^s\ln x\left[1 - \frac{x^2}{(s+2)^2} + \frac{x^4}{(s+2)^2(s+4)^2} - \ldots\right]\\ &\quad + x^s\left[-\frac{x^2}{(s+2)^2}\left(\frac{-2}{s+2}\right) + \frac{x^4}{(s+2)^2(s+4)^2}\left(\frac{-2}{s+2} - \frac{2}{s+4}\right)\right.\\ &\qquad\left. - \frac{x^6}{(s+2)^2(s+4)^2(s+6)^2}\left(\frac{-2}{s+2} - \frac{2}{s+4} - \frac{2}{s+6}\right) + \ldots\right]\end{aligned}$$

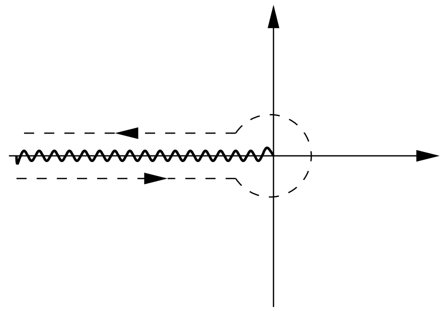

Figure 4: Integration contour for Problem 6.

The first term is, of course, $y(x,s)\ln x$; and setting $s=0$ in both terms yields

$$J_0(x)\ln x + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 - \ldots$$

$$\boxed{A = \frac{1}{4} \qquad B = -\frac{3}{128} \qquad C = \frac{11}{13824}}$$

Note: While this is perhaps the shortest way to solve this problem, it is by no means the only one. It is also possible to obtain this solution by using the Wronskian. One can also show determine the coefficients A, B, and C by simply plugging in the desired solution as a guess, although this method lacks a certain something.

6. Show that

$$J_\nu(x) = \frac{1}{2\pi i}\int_C \exp\left[\frac{x}{2}\left(w - \frac{1}{w}\right)\right] w^{-\nu-1}\,\mathrm{d}w,$$

where C is the contour shown in Figure 4, satisfies Bessel's equation.

We have

$$\frac{\mathrm{d}}{\mathrm{d}x}J_\nu(x) = \frac{1}{4\pi i}\int_C \exp\left[\frac{x}{2}\left(w - \frac{1}{w}\right)\right]\left(w - \frac{1}{w}\right) w^{-\nu-1}\,\mathrm{d}w$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}J_\nu(x) = \frac{1}{8\pi i}\int_C \exp\left[\frac{x}{2}\left(w - \frac{1}{w}\right)\right]\left(w - \frac{1}{w}\right)^2 w^{-\nu-1}\,\mathrm{d}w$$

Thus,

$$x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} J_\nu(x) + x \frac{\mathrm{d}}{\mathrm{d}x} J_\nu(x) + (x^2 - \nu^2) J_\nu(x)$$
$$= \frac{1}{2\pi i} \int_C \exp\left[\frac{x}{2}\left(w - \frac{1}{w}\right)\right] \left(\frac{x^2}{4}\left(w - \frac{1}{w}\right)^2 + \frac{x}{2}\left(w - \frac{1}{w}\right) + x^2 - \nu^2\right) w^{-\nu-1}\,\mathrm{d}w$$

But

$$\exp\left[\frac{x}{2}\left(w - \frac{1}{w}\right)\right] \left(\frac{x^2}{4}\left(w - \frac{1}{w}\right)^2 + \frac{x}{2}\left(w - \frac{1}{w}\right) + x^2 - \nu^2\right) w^{-\nu-1}$$
$$= \frac{\mathrm{d}}{\mathrm{d}w}\left\{\left[\frac{x}{2}\left(w + \frac{1}{w}\right) + \nu\right] \exp\left[\frac{x}{2}\left(w - \frac{1}{w}\right)\right] w^{-\nu}\right\}$$

and so the integral becomes

$$\mathcal{L}[J_\nu(x)] = \frac{1}{2\pi i}\left\{\left[\frac{x}{2}\left(w + \frac{1}{w}\right) + \nu\right] \exp\left[\frac{x}{2}\left(w - \frac{1}{w}\right)\right] w^{-\nu}\right\}_{w\to-\infty-i\epsilon}^{w\to-\infty+i\epsilon},$$

where $\mathcal{L}$ is the Bessel operator. As $w \to -\infty \pm i\epsilon$, we have

$$\lim_{w\to-\infty\pm i\epsilon} w^{-\nu} = \exp\left[-\nu(\pm i\pi + \ln|w|)\right] = e^{\mp i\pi\nu}|w|^{-\nu}$$

and so we have

$$\mathcal{L}[J_\nu(x)] = \frac{x}{\pi} \lim_{|w|\to\infty} \sin(\nu\pi) e^{-x|w|/2} |w|^{1-\nu}$$

(noting that the w^{-1} terms are negligible in this limit and that $w \to -|w|$ there.) This quantity will indeed vanish, assuming that one of the following three conditions holds:

- $\nu \in \mathbb{Z}$, in which case the sine term will vanish;
- $\Re(x) > 0$, in which case the exponential will decay as $|w| \to \infty$; or
- $\Re(x) = 0$ and $\nu > 1$, in which case the $|w|^{1-\nu}$ term will decay.