Problem Set 6: Solutions

Physics 330

M. Seifert

Due Date: November 21, 2007

1. (MW 6-2) Let U be a unitary matrix and let $\vec{x}_1$, $\vec{x}_2$ be two eigenvectors of U with eigenvalues λ_1, λ_2 respectively.

(*a*) Show that $|\lambda_1| = 1$.

We have $U\vec{x}_1 = \lambda_1 \vec{x}_1$, and so $(U\vec{x}_1)^\dagger = \vec{x}_1^\dagger U^\dagger = \lambda_1^* \vec{x}_1^\dagger$. Multiplying these two together, we have

$$\begin{aligned} (\vec{x}_1^\dagger U^\dagger)(U\vec{x}_1) &= \lambda_1 \lambda_1^* \vec{x}_1^\dagger \vec{x}_1 \\ \vec{x}_1^\dagger (U^\dagger U)\vec{x}_1 &= |\lambda_1|^2 \vec{x}_1^\dagger \vec{x}_1 \\ \vec{x}_1^\dagger \vec{x}_1 &= |\lambda_1|^2 \vec{x}_1^\dagger \vec{x}_1 \end{aligned}$$

(where we have used the fact that U is unitary in the last step.) Since $\vec{x}_1 \neq 0$, $\vec{x}_1^\dagger \vec{x}_1 > 0$, and thus we conclude that $|\lambda_1| = 1$. (An identical proof holds, of course, for λ_2.) ☐

(*b*) Show that if $\lambda_1 \neq \lambda_2$, $\vec{x}_1^\dagger \vec{x}_2 = 0$.

We have $U\vec{x}_2 = \lambda_2 \vec{x}_2$ and (as in the previous part) $\vec{x}_1^\dagger U^\dagger = \lambda_1^* \vec{x}_1^\dagger$. Multiplying theses two together as we did above yields

$$\begin{aligned} \vec{x}_1^\dagger U^\dagger U \vec{x}_1 &= \lambda_1^* \lambda_2 \vec{x}_1^\dagger \vec{x}_2 \\ (1 - \lambda_1^* \lambda_2)\vec{x}_1^\dagger \vec{x}_2 &= 0 \end{aligned}$$

We know from the previous part that $\lambda_1^* \lambda_1 = 1$; thus, if $\lambda_1 \neq \lambda_2$, $\lambda_2^* \lambda_1 \neq 1$, and the only way the left-hand side of the above equation can vanish is if $\vec{x}_1^\dagger \vec{x}_2 = 0$. ☐

2. (MW 6-3) Let A and B be Hermitian matrices, and let C and D be unitary.

(*a*) Show that $C^{-1}AC$ is Hermitian.

A matrix is Hermitian iff it equals its Hermitian conjugate. In our case, $(C^{-1}AC)^\dagger = C^\dagger A^\dagger (C^{-1})^\dagger = C^{-1} A (C^\dagger)^\dagger = C^{-1}AC$. So $C^{-1}AC$ is Hermitian. ☐

(b) Show that $C^{-1}DC$ is unitary.

A matrix is unitary iff its inverse is its Hermitian conjugate. In our case, $(C^{-1}DC)^{-1} = C^{-1}D^{-1}C = C^\dagger D^\dagger C = (C^\dagger DC)^\dagger = (C^{-1}DC)^\dagger$. So $C^{-1}DC$ is unitary. □

(c) Show that $i(AB - BA)$ is Hermitian.

We again take the Hermitian conjugate: $(i(AB - BA))^\dagger = -i(B^\dagger A^\dagger - A^\dagger B^\dagger) = -i(BA - AB) = i(AB - BA)$. So $i(AB - BA)$ is Hermitian. □

3. Show that a normal matrix ($N^\dagger N = NN^\dagger \neq \mathbf{1}$) is unitarily diagonalizable.

Given an arbitrary matrix N, define two new matrices A and B:

$$A = \frac{1}{2}(N + N^\dagger) \qquad B = \frac{1}{2i}(N - N^\dagger)$$

By definition, A and B are both Hermitian. We also note that

$$N = A + iB \quad \text{and} \quad N^\dagger = A - iB.$$

The condition that N be normal then becomes

$$\begin{aligned} NN^\dagger - N^\dagger N = [N, N^\dagger] &= 0 \\ [A + iB, A - iB] &= 0 \\ [A, A] - 2i[A, B] + [B, B] &= 0 \\ [A, B] &= 0 \end{aligned}$$

Thus, if N is normal, A and B are commuting Hermitian matrices. This means that they are simultaneously diagonalizable, i.e. there exists a unitary matrix U such that $UAU^\dagger$ and $UBU^\dagger$ are both diagonal. But for this U, we then have

$$UNU^\dagger = UAU^\dagger + iUBU^\dagger$$

which is the sum of two diagonal matrices. Thus, N is unitarily diagonalizable. □

Note: A different method of proof was commonly attempted, which relied on the statement "if D is diagonal and $[D, M] = 0$, then M must also be diagonal." This, unfortunately, is not true: a counterexample would be the two matrices

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad M = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

which commute for arbitrary a, b, c, d, e because of the degeneracy of the entries of D. It's true, however, that M must be block-diagonal and that these blocks must correspond to the different eigenspaces of D; the proof can be salvaged via this observation, although it's a little trickier.

4. Fun with cohomology!

(i) Suppose $Q : V \to V$ is a linear operator on V satisfying $Q^2 = 0$. Show that $\mathrm{Im}(Q)$ is a subspace of $\mathrm{Ker}(Q)$.

We need to show that $\mathrm{Im}(Q)$ is closed under addition and multiplication and that $\mathrm{Im}(Q) \subseteq \mathrm{Ker}(Q)$. The first condition is true for any linear operator: let $\vec{v}_1, \vec{v}_2 \in \mathrm{Im}(Q)$. We wish to show that an arbitrary linear combination $\alpha\vec{v}_1 + \beta\vec{v}_2 \in \mathrm{Im}(Q)$ as well. Since both vectors are in the image of Q, there must be vectors $\vec{w}_1$ and $\vec{w}_2$ such that $Q\vec{w}_1 = \vec{v}_1$ and $Q\vec{w}_2 = \vec{v}_2$. But then by the linearity of Q, $Q(\alpha\vec{w}_1 + \beta\vec{w}_2) = \alpha Q\vec{w}_1 + \beta Q\vec{w}_2 = \alpha\vec{v}_1 + \beta\vec{v}_2$. Thus, $\alpha\vec{v}_1 + \beta\vec{v}_2 \in \mathrm{Im}(Q)$, and $\mathrm{Im}(Q)$ is closed under addition & scalar multiplication.

The second condition is a consequence of Q being nilpotent. Suppose $\vec{v} \in \mathrm{Im}(Q)$. This means that there exists a vector $\vec{w} \in V$ such that $Q\vec{w} = \vec{v}$. But then $Q\vec{v} = Q^2\vec{w} = 0$, and thus $\vec{v} \in \mathrm{Ker}(Q)$. We conclude that $\mathrm{Im}(Q)$ is a subspace of $\mathrm{Ker}Q$. ☐

(ii) Let

$$Q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

acting on $\mathbb{C}^2$. Compute $H^*(Q) = \mathrm{Ker}(Q)/\mathrm{Im}(Q)$.

The operator Q acting on an arbitrary vector $\vec{v} \in \mathbb{C}^2$ yields

$$Q \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_2 \\ 0 \end{bmatrix}$$

The kernel of Q is thus all vectors which are mapped to zero; this means that $\alpha_2 = 0$, or

$$\mathrm{Ker}(Q) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Meanwhile, the image of Q is easily seen to be

$$\mathrm{Im}(Q) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Thus, $\mathrm{Ker}(Q) = \mathrm{Im}(Q)$, and $H^*(Q) = \{0\}$.

(iii) Suppose we have a chain of vector spaces $V_0, V_1, \ldots, V_{n-1}$ and linear operators

$$Q_i : V_i \to V_{i+1} \qquad i = 0, 1, \ldots, n-1$$

satisfying three conditions:

- $Q_{i+1} \circ Q_i = 0$ for $i \in \{0, 1, \ldots, n-1\}$;
- $\mathrm{Ker}(Q_0) = \{0\}$; and
- $\mathrm{Coker}(Q_{n-1}) = \{0\}$.

Define

$$\begin{aligned} H^i &\equiv \mathrm{Ker}(Q_i)/\mathrm{Im}(Q_{i-1}) \qquad i = 1, 2, \ldots, n-1 \\ H^0 &\equiv \mathrm{Ker}(Q_0) \end{aligned}$$

Show that the vector spaces H^i are well-defined.

Since the kernel of any linear operator is a vector space, H^0 is well-defined as a vector space. For the remaining H^i, we need to show that the space $\mathrm{Im}(Q_{i-1})$ is a subspace of $\mathrm{Ker}(Q_i)$. That $\mathrm{Im}(Q_{i-1})$ is closed follows from the same argument used in part (i). To show that $\mathrm{Im}(Q_{i-1})$ is contained in $\mathrm{Ker}(Q_i)$, we note that if $\vec{v} \in \mathrm{Im}(Q_{i-1}) \subset V_i$, then there necessarily exists a $\vec{w} \in V_{i-1}$ such that $Q_{i-1}\vec{w} = \vec{v}$. Applying Q_i to both sides of this equation then yields $Q_i\vec{v} = (Q_i \circ Q_{i-1})\vec{w} = 0$ by the first condition above. Thus, $\mathrm{Im}(Q_{i-1})$ is a subspace of $\mathrm{Ker}(Q_i)$, and the vector spaces H^i are well-defined. □

(iv) Show that

$$\sum_{i=0}^{n} t^i \dim(V_i) - \sum_{i=0}^{n-1} t^i \dim(H^i) = (1+t)Q(t)$$

where $Q(t)$ is a polynomial with non-negative coefficients.

We first cite some basic facts about the dimensions of the vector spaces we're using. The dimension of a quotient space $C = A/B$ is given by $\dim(C) = \dim(A) - \dim(B)$; thus, for $i = 1, \ldots, n-1$, we will have

$$\dim(H^i) = \dim(\mathrm{Ker}(Q_i)) - \dim(\mathrm{Im}(Q_{i-1}))$$

We also know that for an arbitrary linear map between two vector spaces $f : A \to B$, we will have $\dim(A) = \dim(\mathrm{Ker}(f)) + \dim(\mathrm{Im}(f))$; in our case, then,

$$\dim(V_i) = \dim(\mathrm{Ker}(Q_i)) + \dim(\mathrm{Im}(Q_i)).$$

With these facts in mind, we can solve the problem. We have

$$\begin{aligned} &\sum_{i=0}^{n} t^i \dim(V_i) - \sum_{i=0}^{n-1} t^i \dim(H^i) \\ &\qquad = t^n \dim(V_n) + \sum_{i=0}^{n-1} t^i \left[\dim(\mathrm{Ker}(Q_i)) + \dim(\mathrm{Im}(Q_i))\right] \\ &\qquad + \sum_{i=1}^{n-1} t^i \left[\dim(\mathrm{Im}(Q_{i-1})) - \dim(\mathrm{Ker}(Q_i))\right] - t^0 \dim(\mathrm{Ker}(Q_0)) \end{aligned}$$

All of the terms containing the dimension of the kernel of Q_i cancel out.

We can also rewrite the second sum, yielding

$$\begin{aligned}\sum_{i=0}^{n} t^i \dim(V_i) - \sum_{i=0}^{n-1} t^i \dim(H^i) \\ &= t^n \dim(V_n) + \sum_{i=0}^{n-1} t^i \dim(\operatorname{Im}(Q_i)) + \sum_{i=0}^{n-2} t^{i+1} \dim(\operatorname{Im}(Q_i)) \\ &= t^n \dim(V_n) + (1+t)\left[\sum_{i=0}^{n-1} t^i \dim(\operatorname{Im}(Q_i))\right] - t^n \dim(\operatorname{Im}(Q_{n-1}))\end{aligned}$$

(In the last step, we have combined the two sums and included the last term to account for the different limits of summation.) But the cokernel of Q_{n-1}, defined as $V_n/\operatorname{Im}(Q_{n-1})$, is trivial; in other words, $\dim(V_n) = \dim(\operatorname{Im}(Q_{n-1}))$, and we conclude that

$$\sum_{i=0}^{n} t^i \dim(V_i) - \sum_{i=0}^{n-1} t^i \dim(H^i) = (1+t)\left[\sum_{i=0}^{n-1} t^i \dim(\operatorname{Im}(Q_i))\right]$$

The sum in square brackets is then our polynomial $Q(t)$; note that it has positive coefficients, as desired. ☐

(v) Show that

$$\sum (-1)^i \dim(V_i) = \sum (-1)^i \dim(H^i)$$

Setting $t = -1$ in the formula derived in the previous part causes $(1+t)Q(t)$ to vanish; rearranging the equation then yields the desired result. ☐