

Physics 445

Problem Set 1 solutions

Eduard Antonyan

1 Finite gauge transformations

Let ψ be a field transforming in representation r , i.e.

$$\psi \rightarrow \psi' = V(x)\psi = e^{i\alpha^a(x)t_r^a}\psi \quad (1)$$

Then the gauge field should transform in such a way that $D_\mu\psi \rightarrow V(x)D_\mu\psi$, so that the Lagrangian remains invariant under the gauge transformations.

We thus require

$$D'_\mu\psi' = (\partial_\mu - igA'_\mu{}^a t_r^a)V(x)\psi = V(x)D_\mu\psi \quad (2)$$

which implies then after rearranging

$$A'_\mu = V(x) \left(A_{\mu a} + \frac{i}{g} \partial_\mu \alpha^a \right) V^\dagger(x) \quad (3)$$

with $A_\mu \equiv A_\mu^a t_r^a$.

2 Grungy algebra

(a) It should be clear that the number of $N \times N$ hermitian traceless matrices is $N^2 - 1$, which is equal to 8 for $SU(3)$.

(b) As the problem title implies - this is just grungy algebra. Let us check the answer for just one set of f 's. A quick calculation shows that $f^{123} = f^{231} = f^{312} = -f^{132} = -f^{321} = -f^{213} = 1$.

(c) With this normalization we can check that $\text{tr}(t^a t^a) = 1/2$ for any a , so that $C(r) = 1/2$.

(d) Multiplying the matrices together and summing up we get

$$\sum t_r^a t_r^a = \frac{4}{3} \mathbf{1} \quad (4)$$

so that $C_2(r) = \frac{4}{3}$. Since $d(r) = 3$ and $d(G) = 8$ we see that

$$d(r)C_2(r) = d(G)C(r) = 4 \quad (5)$$

3 Casimir invariants

(a) As we know for $SU(2)$ spin and C_2 are related via $C_2 = j(j+1)$. Then for a single $SU(2)$

$$C \cdot 3 = j(j+1)(2j+1) \quad (6)$$

where we used $d(SU(2)) = 3$ and $d(j) = 2j+1$. Summing over the representations we then have for a general group

$$3C(r) = \sum_i j_i(j_i+1)(2j_i+1) \quad (7)$$

(b) Ignoring the singlets which have $j = 0$ we have

$$3C(N) = \frac{1}{2} \frac{3}{2} 2 = \frac{3}{2} \quad (8)$$

so that $C(N) = \frac{1}{2}$. For the adjoint we have the following decomposition

$$\underline{N^2 - 1} \rightarrow (\underline{2} + (N - 2)\underline{1})^2 \rightarrow \underline{2} \times \underline{2} + 2(N - 2)\underline{2} \times \underline{1} \rightarrow \underline{3} + 2(N - 2)\underline{2} \quad (9)$$

where we dropped the singlets. Therefore

$$3C(G) = 1 \cdot 2 \cdot 3 + 2(N - 2)\frac{1}{2} \cdot 2 = 3N \quad (10)$$

(c) The symmetric and antisymmetric representations of $SU(N)$ decompose under $SU(2)$ subgroup representations as follows (ignoring the singlets)

$$\begin{aligned} \underline{S} &= \frac{N(N+1)}{2} \rightarrow \underline{3} + (N-2)\underline{2} \\ \underline{A} &= \frac{N(N-1)}{2} \rightarrow \underline{1} + (N-2)\underline{2} \end{aligned} \quad (11)$$

We therefore have $C(S) = \frac{N+2}{2}$ and $C(A) = \frac{N-2}{2}$. Using the fact that $d(G) = N^2 - 1$ we get the quadratic Casimirs

$$\begin{aligned} C_2(S) &= \frac{(N-1)(N+2)}{N} \\ C_2(A) &= \frac{(N+1)(N-2)}{N} \end{aligned} \quad (12)$$

Since $\underline{N} \times \underline{N} = \underline{S} + \underline{A}$ using equation 15.100 we get

$$\text{tr}(t_{N \times N}^a)^2 = [C_2(N) + C_2(N)]d(N)d(N) = N(N^2 - 1) \quad (13)$$

On the other hand from the above calculations

$$C_2(S)d(S) + C_2(A)d(A) = \frac{1}{2}(N^2 - 1)(N + 2) + \frac{1}{2}(N^2 - 1)(N - 2) = N(N^2 - 1) \quad (14)$$

4 Coulomb potential

(a) Introducing an auxiliary field $J^\mu(x)$ in the action the Wilson loops is given by

$$\langle U_P \rangle = \frac{1}{Z_0} \int \mathcal{D}A e^{i \int (L + J^\mu A_\mu)} \quad (15)$$

In Feynmann gauge

$$\int d^4x L = \frac{1}{2} \int d^4x A_\mu(x) (\partial^2 g^{\mu\nu}) A_\nu(x) \quad (16)$$

Shifting the gauge field to complete the square in the action: $A'_\mu(x) = A_\mu(x) - i \int d^4y D_{F\mu\nu}(x-y) J^\nu(y)$ similar to equation 9.39 we get

$$\langle U_P \rangle = \exp \left[-\frac{e^2}{2} \oint_P dx^\mu \oint_P dy^\nu D_{F\mu\nu}(x-y) \right] \quad (17)$$

One can also compute the propagator

$$D_{F\mu\nu}(x-y) = \int \frac{d^4k}{2(\pi)^4} \frac{e^{-ik(x-y)}}{k^2 + i\epsilon} (-ig_{\mu\nu}) = \frac{g_{\mu\nu}}{4\pi^2(x-y)^2} \quad (18)$$

(b) We now want to compute $I \equiv \oint_P dx^\mu \oint_P dy^\nu \frac{g_{\mu\nu}}{(x-y)^2}$ along a rectangular path. Since we want $R \ll T$ we ignore the integral along the spatial direction, so that

$$\begin{aligned} I &\approx \int dx^0 \int dy^0 \frac{1}{(x-y)^2} = \int_0^T dx^0 \int_0^T dy^0 \frac{1}{(x^0 - y^0)^2} + \int_0^T dx^0 \int_T^0 dy^0 \frac{1}{(x^0 - y^0)^2 - R^2} + \\ &\int_T^0 dx^0 \int_0^T dy^0 \frac{1}{(x^0 - y^0)^2 - R^2} + \int_T^0 dx^0 \int_T^0 dy^0 \frac{1}{(x^0 - y^0)^2} \end{aligned} \quad (19)$$

Calculating the R -dependant piece we get

$$\int_0^T dx^0 \int_0^T dy^0 \frac{1}{(x^0 - y^0)^2 - R^2} = \frac{\pi i T}{R} \quad (20)$$

so that $\langle U_P \rangle \propto e^{\frac{iT e^2}{4\pi R}}$. Comparing with the general expression we get $E(R) = -\frac{e^2}{4\pi R}$, the familiar Coulomb potential.
(c) The only difference for the non-Abelian gauge field (if we only consider the contribution from quadratic terms in the action) throughout is the change from e^2 to $g^2 t_r^a t_r^b \delta^{ab} = C_2(r) g^2$. So that $V(R) = -\frac{g^2 C_2(r)}{4\pi R}$.