

Physics 445

Problem Set 2 solutions

Eduard Antonyan

1 Spin waves

(a) This calculation can be reduced to calculating the generating functional $Z[J]$ for a free scalar field:

$$\langle T e^{i\phi(x)} e^{-i\phi(y)} \rangle = \frac{1}{Z_0} \int \mathcal{D}\phi e^{i \int d^4z \mathcal{L}_0 \phi} e^{i\phi(x)} e^{-i\phi(y)} = \frac{Z[J]}{Z_0} \quad (1)$$

with $J = \delta^4(z-x) - \delta^4(z-y)$. Borrowing the result from P&S 9.39 we then have

$$\langle T e^{i\phi(x)} e^{-i\phi(y)} \rangle = e^{D(x-y) - D(0)} \quad (2)$$

(b) Since we need translational symmetry we can only have derivatives of ϕ appearing in the Lagrangian. Now in d dimensions $[\phi] = d/2 - 1$. Suppose now we have a term in the Lagrangian which has m derivatives and n ϕ 's (with $n \leq m$ since we don't want e.g. $\partial(\phi^2)$ terms). Such a term would have mass dimension $n(d/2 - 1) + m$ which should be less than or equal to d to be renormalizable. Combining that with $n \leq m$ we get

$$\frac{nd}{2} \leq d, \text{ or } n \leq 2 \quad (3)$$

The $n = 1$ term is a pure derivative and so we are left with just the free Lagrangian - $n = 2$ and $m = 2$!

(c) Using the result of part (a) we have

$$\langle s(x)s^*(0) \rangle = A^2 e^{D(x) - D(0)} \quad (4)$$

where

$$D(x) = \frac{1}{\rho} D_F(x) = \frac{1}{\rho} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2} e^{-ipx} = \frac{\Gamma(d/2 - 1)}{4\rho\pi^{d/2}} |x|^{2-d} \quad (5)$$

We can adjust A to give a sensible result by absorbing all the divergences into it, in particular the divergences coming from $D(0)$, by cutting off $D(0)$ at one atomic spacing and then absorbing everything into A .

Now plugging $D(x)$ back we find in $d > 2$ that as $|x| \rightarrow \infty$, $\langle s(x)s^*(0) \rangle \rightarrow 1$ so that the symmetry is spontaneously broken. For $d \leq 2$ we find $\langle s(x)s^*(0) \rangle \rightarrow 0$ and the symmetry remains unbroken.

2 The Gross-Neveu model

(a) Under the transformation $\psi_i \rightarrow \gamma^5 \psi_i$, we have $\bar{\psi}_i \rightarrow -\bar{\psi}_i \gamma^5$, so that $\bar{\psi}_i \psi_i \rightarrow -\bar{\psi}_i \psi_i$ making a mass term non-invariant. The kinetic term remains invariant because commuting γ^5 through the gamma matrix in the kinetic term picks up an additional minus sign.

(b) In 2 dimensions $[\psi] = 1/2$. The coupling constant is then dimensionless and so the theory is renormalizable.

(c) Let us add the following quadratic term to the Lagrangian, i.e. a constant to the path integral

$$-\frac{1}{2g^2} (\sigma + g^2 \bar{\psi}_i \psi_i)^2 \quad (6)$$

Expanding we get the new Lagrangian

$$\bar{\psi}_i i \not{\partial} \psi_i - \sigma \bar{\psi}_i \psi_i - \frac{1}{2g^2} \sigma^2 \quad (7)$$

(d) Integrating over the fermion fields we'll get a determinant (since the action is quadratic) which will give an effective potential for σ . The path

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\int d^2x \mathcal{L}} = \int \mathcal{D}\sigma e^{i\int d^2x \left(-\frac{\sigma^2}{2g^2}\right)} [\det(i\partial - \sigma)]^N \quad (8)$$

We can compute the determinant by going to momentum space

$$\det(\not{k} - \sigma) = \det \begin{pmatrix} -\sigma & -i(k^0 + k^1) \\ i(k^0 - k^1) & \sigma \end{pmatrix} = k^2 - \sigma^2 \quad (9)$$

so that

$$\log \det(i\partial - \sigma) = VT \int \frac{d^2k}{(2\pi)^2} \log(k^2 - \sigma^2) \quad (10)$$

We can use dimensional regularization and minimal subtraction scheme to compute the integral

$$[\det(i\partial - \sigma)]^N = \exp \left[-iNVT \frac{\sigma^2}{4\pi} \log \left(\frac{\sigma^2}{M^2} \right) \right] \quad (11)$$

where M is the mass scale introduced by the $\overline{\text{MS}}$ scheme.

To conclude the effective potential for σ is given by

$$V_{eff} = N \left[\frac{\sigma^2}{2g^2N} + \frac{\sigma^2}{4\pi} \log \frac{\sigma^2}{M^2} \right] \quad (12)$$

(e) Solving for the extrema of the potential we find three solutions: $\sigma = 0$ and $\sigma^2 = M^2 e^{-(1 + \frac{2\pi}{g^2N})}$. It's easy to check that the first solution is a local maximum, while the last two are minima. Therefore the $\sigma \leftrightarrow -\sigma$ symmetry is spontaneously broken by the vacuum, which means that the fermions acquire dynamical masses of the order of the expectation value of σ . Moreover it easy to see that working with a different renormalization scheme will only affect an overall constant in front of the symmetry breaking solution and can just be reabsorbed into M .

(f) Consider a diagram with L loops, V vertices and P propagators. Each vertex contributes a factor of N and each propagator a factor $1/N$. Thus we get an overall factor of $N^{V-P} = 1/N^{L-1}$ using $P - V + 1 = L$. Therefore the higher loop contributions are suppressed by a factor of $1/N$ and the above results are exact in the $N \rightarrow \infty$ limit.

3 Arnowitt-Fickler gauge

To do the Fadeev-Popov quantization in the $A^{3a} = 0$ gauge we take $G(A) = A^{3a} - \omega^a$. The Fadeev-Popov procedure then gives the Lagrangian

$$\mathcal{L}_{FP} = \mathcal{L}_{YM} - \bar{c} \left(\frac{1}{g} D_3 \right) c - \frac{1}{2\chi} (A_3^a)^2 \quad (13)$$

The Feynmann rules remain unchanged except for the propagator which receives a correction from the extra term. Introducing a vector $n^\mu = (0, 0, 0, 1)$ we can write the additional terms more covariantly as $(n \cdot A)^2$. Then it's easy to see that the propagator becomes

$$-\frac{i}{k^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{(n \cdot k)^2} - \frac{k^\mu n^\nu + k^\nu n^\mu}{k \cdot n} \right) \quad (14)$$

There are no kinetic terms for the ghosts and thus no propagating ghosts.

The gauge choice $A^{3a} = 0$ restricts the polarization vectors to be perpendicular to the z axis. We still have the residual gauge freedom of shifting by a gauge parameter independent of z , which we can use to remove one more polarization. Thus we have two positive metric degrees of freedom.

4 BRST

Plugging in the BRST transformations

$$\begin{aligned} \delta A_\mu &= \epsilon \partial_\mu c & \delta \psi &= ig\epsilon c \psi & \delta \bar{\psi} &= -ig\bar{\psi} \epsilon c \\ \delta c &= 0 & \delta \bar{c} &= -\frac{\epsilon}{\chi} \partial^\mu A_\mu \end{aligned} \quad (15)$$

we see that the Lagrangian is invariant (note that part of the BRST transformations is just a gauge transformation). Going through the Noether procedure we find the corresponding current

$$j^\mu = -F^{\mu\nu} \epsilon \partial_\nu c - \partial^\nu A_\nu \epsilon \partial^\mu c \quad (16)$$

with the current given by the integral of j^0 . In terms of the creation/annihilation operators

$$\begin{aligned} c(x) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} [c(p)e^{ipx} + c^\dagger(p)e^{-ipx}] \\ \bar{c}(x) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} [\bar{c}(p)e^{ipx} + \bar{c}^\dagger(p)e^{-ipx}] \end{aligned} \quad (17)$$

$$A_\mu(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} [a_\mu(p)e^{ipx} + a_\mu^\dagger(p)e^{-ipx}] \quad (18)$$

we get the BRST charge

$$Q = -\epsilon \int \frac{d^3 p}{(2\pi)^3} (p^\nu a_\nu^\dagger(p)c(p) + p^\nu a_\nu(p)c^\dagger(p)) \quad (19)$$

Imposing the usual (anti-)commutation relations: $[a^\mu(p), a^{\nu\dagger}(p')] = (2\pi)^3 g^{\mu\nu} \delta^3(p - p')$ etc., we find the following relations for Q

$$\begin{aligned} [Q, a^\mu(p)] &= -p^\mu c(p), \quad [Q, a^{\mu\dagger}(p)] = p^\mu c^\dagger(p) \\ \{Q, \bar{c}(p)\} &= p^\mu a_\mu(p), \quad \{Q, \bar{c}^\dagger(p)\} = p^\mu a_\mu^\dagger(p) \\ \{Q, c(p)\} &= \{Q, c^\dagger(p)\} = 0 \end{aligned} \quad (20)$$

5 Differential Forms

(i) The antisymmetry of forms allows us to write F in terms of A as $F = dA$.

(ii) The Maxwell equations are

$$\begin{aligned} \partial_{[\mu} F_{\nu\lambda]} &= 0 \\ \partial_\mu F^{\mu\nu} &= -4\pi j^\nu \end{aligned} \quad (21)$$

It's easy to see that in terms of forms these can be written as (note that the first one - the Bianchi identity - is derived by simply noting that $d^2 = 0$ and thus is there by definition of F and not by the equations of motion)

$$\begin{aligned} dF &= 0 \\ d * F &= 4\pi * J \end{aligned} \quad (22)$$

(iii) We can generalize the above to the non-Abelian case. The field strength becomes $F = dA - igA \wedge A$. The Bianchi identity then becomes $dF - ig(A \wedge F - F \wedge A) = 0$, while the equation of motion is given by

$$d * F - ig(A \wedge *F - *F \wedge A) = g * J \quad (23)$$

6 Scalar field with non-Abelian charge

(a) The Feynmann rules can be read off from the Lagrangian. We have a standard kinetic term, so we have the usual propagator. The three point vertex (scalar i, p - scalar j, p' - gauge μ, a) is given by $ig(p + p')^\mu t_{ij}^a$ and the four point vertex is given by (scalar - scalar - gauge - gauge) $ig^2 g^{\mu\nu} \{t^a, t^b\}_{jk}$.

(b) The computations are fairly lengthy but straightforward and we will just present the key point results here.

There are 2 diagrams contributing to the scalar field propagator at the one-loop level (one with a gauge rainbow and one with a gauge bubble). Using dimensional regularization we can see that the bubble diagram actually gives no

contribution (introduce a mass term for the gauge boson and then take it to zero to see that). The second diagram gives a non-zero contribution to the counterterm $\delta_\phi = \frac{g^2 C_2(r)}{4\pi^2 \epsilon}$.

There are 5 diagrams for the gauge boson propagator (scalar, ghost and gauge loops; gauge and scalar bubbles). The gauge loop + gauge bubble + ghost loop part is calculated in P&S and gives

$$i(k^2 g^{\mu\nu} - k^\mu k^\nu) \delta^{ab} \frac{-g^2}{16\pi^2} (-5/3) C_2(G) \frac{2}{\epsilon} \quad (24)$$

The scalar bubble gives zero as in the gauge propagator case (we take scalar mass to zero since we're only interested in UV divergences), while the scalar loop diagram gives

$$-ig^2(k^2 g^{\mu\nu} - k^\mu k^\nu) \frac{1}{24\pi^2 \epsilon} C(r) \delta^{ab} \quad (25)$$

Combining together we get the counterterm $\delta_A = g^2 \frac{5C_2(G) - C(r)}{24\pi^2 \epsilon}$.

Next we consider the renormalization of the coupling constant by looking at the three point diagrams at one loop which are even lengthier (and are very similar to the triangular diagrams that one computes for the beta function of non-Abelian gauge theory). Skipping right to the end the counterterm is given by

$$\delta_g = -\frac{g^3 \mu^{\epsilon/2}}{8\pi^2 \epsilon} [C_2(G) - 2C_2(r)] \quad (26)$$

The rest is the easy part. Using $const = g_0 = Z_A^{-1/2} Z_\phi^{-1} (g\mu^{\epsilon/2} + \delta_g)$ and $Z_\phi = 1 + \delta_\phi$, $Z_A = 1 + \delta_A$ we can calculate the beta function

$$\beta = \frac{\partial g}{\partial(\log \mu)} = -g \frac{\epsilon}{2} \left[1 + \frac{g^2}{24\pi^2 \epsilon} (11C_2(G) - C(r)) \right] + \dots = -\frac{g^3}{(4\pi)^2} \left(\frac{11}{3} C_2(G) - \frac{1}{3} C(r) \right) \quad (27)$$