Physics 445
Problem Set 3 solutions

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1 Supersymmetry

(i) Let us look at the variation of each term:

$$\begin{array}{l}\delta(\phi^*\partial^2\phi) = \sqrt{2}(\bar{\psi}\bar{\zeta}\partial^2\phi + \phi^*\zeta\partial^2\psi) \\ \delta(-|\lambda|^2|\phi|^4) = -2^{3/2}|\lambda|^2(|\phi|^2\zeta\psi\phi^* + |\phi|^2\phi\bar{\psi}\bar{\zeta} \\ \delta(-\lambda\psi\psi\phi) = -i2^{3/2}\lambda^*(\zeta\sigma^\mu\bar{\psi})(\partial_\mu\phi^*)\phi^* + 2^{3/2}|\lambda|^2\bar{\zeta}\bar{\psi}\phi|\phi|^2 \\ \delta(-i\psi\sigma^\mu(\partial_\mu\bar{\psi})) = -\sqrt{2}[\zeta(\partial^2\psi)\phi^* + \bar{\psi}\bar{\zeta}(\partial^2\phi)] - 2^{3/2}i[\lambda\bar{\zeta}\bar{\sigma}^\mu\psi(\partial_\mu\phi)\phi + \lambda^*(\bar{\psi}\bar{\sigma}^\mu\zeta)(\partial_\mu\phi^*)\phi^*] + \text{total derivative}\end{array} \tag{1}$$

Combining all together we see that the Lagrangian remains invariant up to a total derivative.
(ii) Let us expand the Lagrangian around a classical background $\phi = \phi_0 + \delta\phi$ and keeping terms which are at most quadratic in fluctuations. The correction to the Lagrangian is then given by

$$\delta\mathcal{L} = \delta\Phi^\dagger \begin{pmatrix} \frac{1}{2}\partial^2 - 2|\lambda|^2|\phi|^2 & -|\lambda|^2\phi^2 \\ -|\lambda|^2\phi^{*2} & \frac{1}{2}\partial^2 - 2|\lambda|^2|\phi|^2 \end{pmatrix} \delta\Phi = \delta\Phi^\dagger M_\phi \delta\Phi \tag{2}$$

where

$$\delta\Phi = \begin{pmatrix} \delta\phi \\ \delta\phi^* \end{pmatrix} \tag{3}$$

Doing a similar expansion for the fermions we find

$$M_\psi = -\frac{i}{2}\begin{pmatrix} \bar{\sigma}^\mu\partial_\mu & -2i\lambda^*\phi^* \\ -2i\lambda\phi & \sigma^\mu\partial_\mu \end{pmatrix} \tag{4}$$

The one-loop correction to the effective action is then given by

$$\Gamma_{corr} = \log\det M_\psi - \frac{1}{2}\log\det M_\phi \tag{5}$$

The determinants are pretty easy to compute in the 1-loop approximation

$$\log\det M_\phi = \log\det\left(-\frac{1}{4}\partial^2 - |\lambda|^2|\phi|^2\right) \approx -8\mathrm{tr}\left(\frac{1}{\partial^2}|\lambda|^2|\phi|^2\right) \tag{6}$$

Where we used $\log\det = \mathrm{tr}\log$ and expanded to first order in λ^2. Doing the same for the fermionic determinant we find that Γ_{corr} is actually zero, so to first order the β-function is zero.

2 Scalar particle decay

The amplitude of a scalar particle decaying into two photons is given by

$$iM = -i8A\epsilon^{\mu\nu\lambda\rho}p_\mu p'_\lambda\epsilon^*_\nu\epsilon^*_\rho \tag{7}$$

Squaring and summing over final state photon polarizations

$$\sum|M|^2 = -128A^2(p^2p'^2 - (p\cdot p')^2) = 128A^2(p\cdot p')^2 \tag{8}$$

since the photon is massless. Using energy-momentum conservation $p \cdot p' = m^2/2$, so that $\sum |M|^2 = 32A^2m^4$ and the tree-level decay rate is

$$\Gamma = \frac{1}{2m}\frac{1}{2}\int \frac{d\Omega_{cm}}{4\pi}\frac{1}{8\pi}\frac{2|\vec{p}|}{E_{cm}}\sum |M|^2 = \frac{A^2m^3}{\pi} \tag{9}$$

(in the center of mass frame $2|\vec{p}| = E_{cm}$).

3 Fun with decays

(a)

$$\frac{1}{2}(j^{\mu 1} + ij^{\mu 2} - j^{\mu 51} - ij^{\mu 52}) = \bar{Q}\gamma^\mu \frac{1-\gamma^5}{2}\tau^1 Q + i\bar{Q}\gamma^\mu \frac{1-\gamma^5}{2}\tau^2 Q = \bar{Q}_L\gamma^\mu(\tau^1+\tau^2)Q_L = \bar{u}_L\gamma^\mu d_L \tag{10}$$

At tree level the decay amplitude is

$$\begin{aligned} -iM(2\pi)^4\delta^4(p-k-q) &= \langle l^+\nu|i\int d^4x \Delta L|\pi^+\rangle = \frac{4iG_F}{\sqrt{2}}\int d^4x \langle l^+\nu|(\bar{l}_L\gamma^\mu\nu_L)(\bar{u}_L\gamma_\mu d_L) + h.c.)|\pi^+\rangle \\ &= \frac{4iG_F}{\sqrt{2}}\int d^4x \bar{u}_L^{s'}(k)\gamma^\mu v_L^s(q)\langle 0|\bar{d}_L\gamma_\mu u_L|\pi^+\rangle e^{ikx}e^{iqx} \end{aligned} \tag{11}$$

Because $j^{\mu a}$ is a vector and is parity invariant, while $|\pi^+\rangle$ is a pseudo scalar and is parity odd, so that $\langle 0|j^{\mu a}|\pi^+\rangle$. Using the identity above then

$$\langle 0|\bar{d}_L\gamma^\mu u_L|\pi^+\rangle = -\frac{1}{\sqrt{2}}\langle 0|j^+_{\mu 5}|\pi\rangle = ip_\mu f_\pi e^{-ipx}\frac{1}{\sqrt{2}} \tag{12}$$

Combining together we have

$$iM = G_F f_\pi \bar{u}(q)\not{p}(1-\gamma_5)v(k) \tag{13}$$

(b) From the amplitude calculated in the first part we have the decay rate

$$\Gamma = \frac{1}{2m_\pi}\int \frac{d\Omega_{cm}}{4\pi}\frac{1}{8\pi}\frac{2|\vec{k}|}{E_{cm}}\sum |M|^2 = \frac{m_\pi m_l^2}{4\pi}(G_F f_\pi)^2\left(1-\frac{m_l^2}{m_\pi^2}\right) \tag{14}$$

where we used $\frac{2|\vec{k}|}{E_{cm}} = 1 - \frac{m_l^2}{m_\pi^2}$. We see that the decay rate goes to zero in the limit of zero lepton mass and

$$\frac{\Gamma(\pi^+ \to e^+\nu)}{\Gamma(\pi^+ \to \mu^+\nu)} = \frac{m_e^2}{m_\mu^2}\frac{(1-m_e^2/m_\pi^2)^2}{(1-m_\mu^2/m_\pi^2)^2} \approx 10^{-4} \tag{15}$$

Plugging in the measured lifetime, pion and muon masses and G_F we find that $f_\pi \approx 90$ MeV.

4 Oh no! It's two loop time

(a) The coupling constant satisfies the renormalization group equation

$$\frac{d}{d(\log(Q/M))}g = \beta(g) = -\frac{b_0}{(4\pi)^2}g^3 - \frac{b_1}{(4\pi)^4}g^5 - \frac{b_2}{(4\pi)^6}g^7 + \dots \tag{16}$$

Changing variables to $y = g^2$, $x = 2\log\frac{Q}{\Lambda}$

$$\frac{dy}{dx} = -C_0y^2 - C_1y^3 + \dots \tag{17}$$

where Λ is a new scale, $C_0 = b_0/(4\pi)^2$ and $C_1 = b_1/(4\pi)^4$. Assuming an initial condition $y(Q=M) = y_0$, to lowest order the solution is

$$y = \frac{y_0}{2C_0y_0\log\frac{Q}{M} + 1} \tag{18}$$

Noting that $\alpha_s = y/4\pi$ and $\alpha_s(Q = M) \equiv \alpha = y_0/4\pi$ and setting Λ to satisfy $1 = y_0(b_0/8\pi^2)\log(M/\Lambda)$ we can rewrite the one loop result as

$$\alpha_s = \frac{4\pi}{b_0 \log \frac{Q^2}{\Lambda^2}} \tag{19}$$

We can also solve the differential equation with y^3 term if we assume that y is small and solve it iteratively. With the above definitions we then get the desired result

$$\alpha_s = \frac{4\pi}{b_0}\left[\frac{1}{\log \frac{Q^2}{\Lambda^2}} - \frac{b_1}{b_0^2}\frac{\log\log \frac{Q^2}{\Lambda^2}}{\left(\log \frac{Q^2}{\Lambda^2}\right)^2}\right] \tag{20}$$

(b) The perturbation series for the e^+e^- annihilation cross section is

$$\sigma = \sigma_0(3\sum_f Q_f^2)\left[1 + \frac{\alpha_s}{\pi} + a_2\left(\frac{\alpha_s}{\pi}\right)^2 + O(\alpha_s^3)\right] \tag{21}$$

Plugging in α_s from part (a) we get

$$\sigma = \sigma_0(3\sum_f Q_f^2)\left[1 + \frac{4}{b_0}\frac{1}{\log \frac{Q^2}{\Lambda^2}} + \ldots\right] \tag{22}$$

We see that the first two terms are then independent of the renormalization prescription (but then we knew this more generally from last quarter).