Black Hole Entropy in $d = 5$

Nausheen R. Shah

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Abstract

I will be discussing Black Holes and their Entropy from the perspective of string theory and supergravity. $d = 10$ supergravity is compactified to $d = 5$. I focus on extremal black holes which are BPS states. Therefore, one can go from the low energy, high coupling limit (supergravity) to high energy, low coupling limit (string perturbation theory) and expect agreement in the microscopic and macroscopic entropy computed in the two limits.

1 Introduction

I have recently become interested in the black hole information paradox. For almost thirty years, since Hawking first proposed the idea of black holes as thermal objects, people have been interested in the contradiction to quantum mechanics that arises due to the non-unitarity of the evaporation process. In recent papers there seems to be a renewed effort to resolve this paradox since there is apparantly some evidence from string theory that actually unitarity is preserved. Therefore I decided to look at said evidence. But before I could do that, I needed to find out how black holes are actually formulated in stringy physics.

Hawking showed that the temperature of the radiation emitted by black holes follows the first law of thermodynamics $dM = T_H dS$, where the entropy is given by $S = \frac{A_H}{4G_N\hbar}$. M is the black hole mass and A_H is the horizon area. When radiating, the black hole mass decreases as does the area of the horizon, but including the entropy of the emitted radiation, the total entropy increases as demanded by the second law of thermodynammics. It has been a mystery for many years what the microscopic degrees of freedom are that give rise to this entropy. String theory is a theory of quantum gravity, so one would expect that we may be able to get a result from it. But the problem has been that black holes involve strong interaction because of their large mass and string theory is defined perturbatively. So only after some non-perturbative tools were available, was this puzzle addressed [3]. We will be looking at charged extremal black holes ($M = Q$) in $d = 5$. Section 2 will briefly go over ten dimensional supergravity. In section 3, we will compactify to $d = 5$ and look at the entropy for these extremal black holes. Section 4 considers the black hole in the low coupling high energy limit where we can apply string perturbation theory. We will find that indeed the entropy computed in these two limits exactly agree. In section 5 we briefly discuss near extremal black holes and Hawking radiation. Section 6 concludes this paper with a brief discussion concluding with some thought about the information paradox.

2 $d = 10$ Supergravity Classical Solutions

I will take a brief look at type II supergravity solutions in ten dimensions [1]. The bosonic part of the action for type IIA supergravity is

$$S = \frac{1}{16\pi G_N^{10}} \int d^{10}x\sqrt{-G}[e^{-2\phi}(R + 4(\bigtriangledown\phi)^2 - \frac{1}{3}H^2) - \alpha' G^2 \qquad (1)$$
$$-\frac{\alpha'}{12}F'^2 - \frac{\alpha'}{288}\epsilon^{\mu_1\ldots\mu_{10}}F_{\mu_1\mu_2\mu_3\mu_4}F_{\mu_5\mu_6\mu_7\mu_8}B_{\mu_9\mu_{10}}]$$

where $G = dA$, $H = dB$, $F = dC$ and $F' = dC + 2A \wedge H$ are the fields associated with each of the differential forms. We are interested in BPS solutions. One of the simplest has only fields in the first three terms excited and is a solution in both type II theories and heterotic string. We start with a $SO(1,1) \times SO(8)$ symmetric ansatz for the metric in string frame,

$$ds^2 = f_s^{-1}(-dt^2 + dx_9^2) + dx_1^2 + ... + dx_8^2 \qquad (2)$$

We allow only the dilaton and B_{09} to be nonzero. For unbroken supersymmetries, the variations of the gravitino and dilatino must vanish. This implies that the dilaton, the metric and the antisymmetric tensor have to be related to each other and have the form

$$ds^2 = f_f^{-1}(-dt^2 + dx_9^2) + dx_1^2 + ... + dx_8^2, \quad (3)$$
$$B_{09} = \frac{1}{2}(f_f^{-1} - 1), \quad (4)$$
$$e^{-2(\phi-\phi_\infty)} = f_f \quad (5)$$

where f_f is a functin of the coordinates $x_1, ..., x_8$. It turns out that the solution is BPS, preserving half the supersymmetries for any function f_f. The equations of motion (related to the closure of the supersymmetry algebra) imply only that f_f must be a harmonic function [2]. Taking

$$f_f = 1 + \frac{Q_f}{r^6}, \quad (6)$$

we get a solution that looks like a long string. There is a singularity at $r = 0$, with a horizon. Here the constant Q_f is arbitrary, but this string carries an "electric" charge under the B field which is an integer multiple of some minimum value. So $Q_f = c_f^{10} m$, where c_f^{10} is the electric charge, and m is an integer.

Since the equations of motion just demand that f_f be harmonic, if we take it to be $f_f = \sum_i c_f^{10}/(\vec{r} - \vec{r_i})^6$ we are describing a collection of strings sitting at positions $\vec{r_i}$ in static equilibrium. The dilatonic and gravitational attraction cancel against the electric repulsion. This superposition principle is a generic property of BPS solutions.

In type II theories, we look for supergravity solutions that describe the long range fields away from the D-brane. So now we will have extended branes of p spatial dimensions. These will carry "electric" charge under the A_{p+1} forms, or "magnetic" charge under the A_{7-p} forms. These solutions will have the form, in string frame,

$$ds^2 = f_p^{-1/2}(-dt^2 + dx_1^2 + ... + dx_p^2) + f_p^{1/2}(dx_{p+1}^2 + ... + dx_9^2), \quad (7)$$
$$e^{-2\phi} = f_p^{\frac{p-3}{2}}, \quad (8)$$
$$A_{0...p} = -\frac{1}{2}(f_p^{-1} - 1). \quad (9)$$

We will only concentrate on the last model writen above. It corresponds to the extremal limit of charged black p-branes with the charge as defined above: Q_p and c_p^{10}, and is related to the minimum charge of the D-brane and can be computed using U-duality.

Another point that should be noted is that all these solutions are boost invariant along the brane, in the sense that they are relativistic branes like the fundamental string. The extremal branes therefore cannot carry momentum in the parallel direction just by rigid motion, but of course they can carry transverse momentum. Longitudinal momentum is carried via oscillations. This leads to BPS solutions preserving 1/4 of the supersymmetries by imposing the additional constraints on the spinors,

$$\epsilon_R^0 = \Gamma^0\Gamma^9\epsilon_R^0, \ \epsilon_L^0 = \Gamma^0\Gamma^9\epsilon_L^0. \tag{10}$$

with momentum in the 9 direction.

3 $d = 5$ Supergravity

Using the superposition principle mentioned above, there is a very nice correspondance between ten dimensional objects and d-dimensional ones. We're interested in solutions where the branes are completely wrapped around the internal dimensions, so that from the point of view of the d-dimensional observer we have a "localized" spherically symmetric solution. When we are in the d-dimensional theory the only way to tell which branes the black hole contains is by looking at the gauge fields it excites. The result is that the d-dimensional solution is given by the same equations as before, but now in terms of d-dimensional harmonic functions.

We will focus on type IIB string theory compactified on T^5. The low energy theory is given by the maximally supersymmetric five dimensional supergravity which is just the dimensional reduction of the $d = 10$ theory. We consider solutions which have Q_5 D-5 branes wrapped around T^5, Q_1 D-1 branes wrapped around S^1, let us pick the 9^{th} direction, and momentum $P = N/R$ along the direction of the D-1 brane. For simplicity we set $\alpha' = 1$. All charges are normalized to be integers. Further, under S-duality $g \to 1/g$.

The solution is given by three harmonic functions, f_5, f_1 and k. We first write the solution in terms of the ten dimensional metric, so that the relationship to (7) becomes clear:

$$ds^2_{str} = f_1^{-1/2} f_5^{-1/2}(-dt^2 + dx_9^2 + k(dt - dx_9)^2) \quad (11)$$
$$+ f_1^{1/2} f_5^{1/2}(dx_1^2 + ... + dx_4^2)$$
$$+ f_1^{1/2} f_5^{-1/2}(dx_5^2 + ... + dx_8^2),$$

$$e^{-2(\phi_{10}-\phi_\infty)} = f_5 f_1^{-1}, \quad (12)$$
$$B'_{09} = \frac{1}{2}(f_1^{-1} - 1), \quad (13)$$

$$H'_{ijk} = (dB')_{ijk} = \frac{1}{2}\epsilon_{ijkl}\partial_l f_5, \quad i,j,k,l = 1,2,3,4 \quad (14)$$

where ϵ_{ijkl} is the flat space epsilon tensor. The three harmonic functions are

$$f_1 = 1 + \frac{c_1^{(5)} Q_1}{x^2}, \; f_5 = 1 + \frac{c_5^{(5)} Q_5}{x^2}, \; k = \frac{c_5^{(5)} N}{x^2} \quad (15)$$

where $x^2 = x_1^2 + ... + x_4^2$ and the c's can again be computed by U-duality. The three independent charges arise as follows: Q_1 is the R-R electric charge coming from B'^{RR}_{09} and it counts the D-1 branes. Q_5 is the magnetic charge for the three form field strength $H_3'^{RR} = dB_2'^{RR}$ which is dual to a two form F_2 in five dimensions so it counts the D-5 branes. The third charge N corresponds to the total momentum along the branes in the 9^{th} direction. It is associated to the five-dimensional Kaluza-Klein fields coming from the G_{09} component of the metric. The D-1 and D-5 branes together with the momentum break some supersymetries as discussed above. So in five dimensions we are left with group $SO(1,1) \times SO(4)_E \times SO(4)_I$ which leaves the ten dimensional solution (11) invariant.

The first $SO(4)_E$ corresponds to rotation in the noncompact spatial directions, and the $SO(4)_I$ corresponds to rotations in the internal directions.

Now if we dimensionally reduce to five dimensions, we get the Einstein metric, $g^5_E = e^{-4\phi_5/3} G^5_{string}$ [2],

$$ds^2 = -\frac{1}{(f_1 f_5(1+k))^{2/3}} dt^2 + (f_1 f_5(1+k))^{1/3}(dx_1^2 + ... + dx_4^2), \quad (16)$$

which describes a five dimensional, extremal, charged, supersymmetric black hole with non-zero horizon area. Calculating the horizon area in this metric we get the entropy

$$S_e = \frac{A_H}{4G_N^5} = 2\pi\sqrt{NQ_1Q_5}. \tag{17}$$

We see that the entropy does not depend on any continous parameters like the coupling constant or the size of the internal dimensions etc. The standard five-dimensional Reissner-Nordstrom solution is recovered when the charges are chosen such that

$$c_P N = c_1 Q_1 = c_5 Q_5 = r_e^2. \tag{18}$$

Tha black hole description given above is well-defined if the curvature everywhere is much smaller than α', since otherwise corrections to the action become important since we are in the high coupling limit.

4 D-Brane Description of Black Holes in Five Dimensions

We are considering type IIB theory compactified on $T^5 = T^4 \times S^1$. As before we have the three charges Q_1, Q_5 and N identified with the number of D-1, D-5 branes and the momentum. Here we take gQ to be small and the radius R_9 to be large.

As mentioned before D-branes are invariant under boosts along directions parallel to them, so they cannot carry momentum by just moving rigidly. Instead we have to consider oscillations to find the D-brane excitations that carry momentum. These are described in terms of massless open strings ending on these branes. In the system we are considering, we have four different kinds of strings (1,1), (5,5), (1,5) and (5,1), where (i, j) corresponds to having one end on the D-i brane and the other end on the D-j brane (we are considering oriented strings so the last two are different).

The BPS mass formula for the whole system implies that the D-brane excitations have to be massless. The excitation energy, which is defined as the total mass of the system minus the mass of the branes, is equal to the momentum. If these excitations are *not* massless, than that would imply that we are contributing more to the energy than to the momentum, so the BPS mass formula will be violated. Therefore we want to find a configuration where we get the maximum number of excited massless strings.

If we take $V^{1/4} << R_9$, then we effectively have a 1+1 field theory which is a (4,4) supersymmetric theory, which is the same amount of supersymmetry as $N = 1$ in six dimensions. T-dualizing in the 1,2,3,4 directions gives us the D-5, D-9 theory, giving us a six dimensional theory on the D-5 brane. Denote by $\alpha, \beta = 0, 1, 2, 3, 4, 9$ the directions along the five brane, $I, J = 5, ..8$ the indices perpendicular to the five branes and $\mu, \nu = 0, ..., 9$ for the full ten dimensional space. For simplicity we will only look at the bosonic part of the Lagrangian and only look at the bosonic part of the supermultiplets. In $N = 1$, $d = 6$ we have two possible supermultiplets: the vector multiplet and the hypermultiplet. The vector contains a six dimensional vector field A_α and its spin 1/2 superpartner. The hypermultiplet contains four scalars and the spin 1/2 superpartners [6]. The ten dimensional vector A_μ describing the (5,5) strings decomposes into a six dimensional vector A_α and a hypermultiplet containing the four scalars A_I. The hypermultiplet represents the transverse motion of the D-5 branes. Both of these fieds are in the adjoint of the $U(Q_1)$ gauge group. The (5,9) and the (9,5) strings are in the fundamental (antifundamental) of $U(Q_1) \times U(Q_5)$, denoted by the hypermultiplets χ_a^B and their complex conjugates, where a and B denote Chan Paton factors. The dimensionally reduced Yang-Mills Action has the form, upto numerical normalizations,

$$S = \frac{1}{g} \int Tr(F_{\alpha\beta} F^{\alpha\beta}) + Tr(F'_{\alpha\beta} F'^{\alpha\beta}) + Tr[(\partial_\alpha A_I + [A_\alpha, A_I])^2] \quad (19)$$
$$+Tr[(\partial_\alpha A'_I + [A'_\alpha, A'_I])^2] + |(\partial_\alpha + A^a_\alpha T^a + A'^a_\alpha T^a)\chi|^2 + \sum D^{a2}_{IJ},$$

The index a runs over both gauge groups, and

$$D^a_I J = f^a_{bc}(A^b_I A^c_J + \frac{1}{2}\epsilon_{IJKL} A^b_K A^c_L) + \chi^\dagger T^a \Gamma_{IJ} \chi. \quad (20)$$

The first term in (20) involves only the (5,5) or the (9,9) strings and the last one the (5,9) or (9,5) strings depending on what gauge group we are considering.

Now T-dualize back to our original D-1, D-5 branes. The supermultiplets are just the dimensional reductions of the six dimensional ones. Some of the components of the six dimensional vector multiplet are now scalars and they represent the motion of the D-1 and D-5 branes in the transverse directions. The motion of the D-1 brane on the five brane, as well as the fivebrane gauge fields transverse to the D-1 brane are hypermultiplets in this picture.

If we give expectation value to the scalars coming from the six dimensional vector field, we are effectively seperating out the D-1 and the D-5 branes [6]. This breaks the gauge symmetry from $U(N)$ to $U(1)^N$, and gives us a small number of massless particles, proportional to $Q_1 + Q_5$. Since we are looking for configurations with the maximum number of massless strings, this is bad.

In fact, we get the maximum number of massless particles by exciting the hypermultiplets which describe the motion of the D-1 branes in the compactified space (χ_a^B). A good way to see this is to realize that the D-1 branes are bound to the D-5 branes, since they are wrapped around S^1 and the D-5 branes wrap the whole compactified space $T^5 = T^4 \times S^1$. So no matter where we localize the D-1 brane on T^4, its still sitting on "top" of the five branes. Therefore giving expectation values to these scalars still gives us massless states.

We will count the number of these massless states two different ways. We can start by looking at the properties of the (1,5) and (5,1) strings. We have Neumann-Neumann boundary conditions in the (X_0, X_9), Dirichlet-Dirichlet boundary conditions for (X_1, X_2, X_3, X_4) and Neumann-Dirichlet boundary conditions for (X_5, X_6, X_7, X_8). The vacuum energy for the worldsheet boson is $E = 4(-1/24 + 1/48)$. Consider first the NS sector for the worldsheet fermion, these are space-time bosons [6]. The vacuum energy will be given by $E = 4(1/24 - 1/48)$. So we have zero vacuum energy. Then we have four periodic fermions on the world sheet in the ND directions. These are invariant under $SO(1,5)$ of the external directions and are two dimensional spinors under $SO(4)_I$. After the GSO projection, we are left with one two-dimensional spinor. There are two possible orientations (1,5) and (5,1) and these can be attached to any of the branes of the two types, so we have a total of $4Q_1Q_5$ different possible states for these strings. The vacuum energy for the R sector is also zero. These are now the space-time fermions. They are invariant under the internal $SO(4)_I$ and spinors under the little group $SO(4)_E$. Again via GSO projection we are left with one two-dimensional spinor. So we get the same number of states as before.

We can also count the total number of hypermultiplets, in the adjoint as well as the fundamental, and then impose constraints to find the truly massless states. The total number of hypermultiplets is $4Q_1^2+4Q_5^2+4Q_1Q_5$. For supersymmetry to be preserved, we must minimize the potential. Therefore we get $3Q_1^2+3Q_5^2$ from setting the D-terms to be zero (20). We also have $Q_1^2+Q_5^2$ gauge equivalent configurations. So the remaining number of massless states is again $4Q_1Q_5$.

As stated before we are effectively in the 1+1 CFT since R_9 is taken to be very large. So we have a total of $4Q_1Q_5$ real scalars and the same amount of Majorana fermions. This theory has central charge $c = (1+1/2)4Q_1Q_5 = 6Q_1Q_5$. Since we are only interested in BPS states, we only want left movers to contribute to the energy, so we have $E = N/R_9$. Thus we are interested in the coeffiecient of the q^N in the expansion of the following partition function [2]

$$Z = (\Pi \frac{1+q^N}{1-q^N})^{4Q_1Q_5} \equiv \sum \Omega(N) q^N, \tag{21}$$

where $\Omega(N)$ is the degeneracy of the states at energy $E = N/R_9$. At large N we can use the Cardy formula

$$\Omega(N) \sim \exp\sqrt{\frac{\pi c E(2\pi R_9)}{3}} = \exp(2\pi\sqrt{\frac{c}{6}ER_9}). \tag{22}$$

Therefore the microscopic entropy is

$$S_{micro} = \log(\Omega(N)) = 2\pi\sqrt{NQ_1Q_5}. \tag{23}$$

This agrees exactly with the black hole result.

4.1 $N \sim Q_1Q_5$

In the previous section we implicitly assumed that the D-1 and D-5 branes were singly wound since we quantized the momentum in units of $1/R_9$. For large $N >> Q_1Q_5$, the entropy is the same no matter how the branes are wound. But for the limit when $N \sim Q_1Q_5$ this breaks down. Let us look at the D-1 brane, ignoring the D-5 brane for the moment. They may be singly wound as assumed in the previous section or they could be connected so as to form a long chain $R' = Q_1 R$ that winds around multiple times. For the former case, the open strings behave like Q_1 species of 1 dimensional particles, with energy spectrum in multiples of $1/R$. In the latter case they behave like a single species of 1 dimensional particles with energy quantized in units of $1/R'$. In other words the system looks like a spectrum of fractional charges. For consistancy, the total charge must add up to be an integer multiple of $1/R$, but it can do so by adding up fractional charges.

Now let us add the D-5 brane back to the picture. We ignore the other four directions that the D-5 brane is wrapped around and just look at its wrapping on S^1 with the D-1 brane. Again they maybe connected together or not. Consider the spectrum of the (1,5) string when both the D-1 and D-5 branes form long chains. The D-1 brane has total length $Q_1 R$ and the D-5 brane has total length $Q_5 R$. Index the open strings with $[i, j]$ labeling which loop of the D-1 and D-5 brane it ends on. If Q_1 and Q_5 are relatively prime the system simulates a single species on a circle of size $Q_1 Q_5 R$. This can be easily checked by looking at the number of times the string would have to cycle to return to its original configuration of $[i, j]$. The situation is slightly more complicated for when the Q's are not prime, but the result is the same.

We can see that this way of wrapping the branes gives the same entropy as before. The open strings have 4 bosonic and 4 fermionic degrees of freedom and carry total momentum N/R. With the branes forming chains, the quantization length is $R' = Q_1 Q_5 R$ and the momentum is quantized in units of $1/R'$. Thus instead of being at level N, it is at level $N' = N Q_1 Q_5$. Instead of the original $Q_1 Q_5$ species we have a single species. The result is

$$S = 2\pi\sqrt{N'} = 2\pi\sqrt{N Q_1 Q_5}. \tag{24}$$

5 Near Extremal Black Holes and Hawking Radiation

The Hawking temperature of the five dimensional extremal black holes is zero so they don't Hawking radiate. To make it non-extremal, we need to add some anti-charges to it. The metric is modified by a non-extremal factor $f = (1 - \frac{r_0^2}{r^2})$ to dt^2 and f^{-1} to

dr^2. So the horizon is shifted to r_o. The easiest way to obtain this metric is by adding momentum modes opposite in direction to the existing ones on X_9. Now the Hawking temperature is proportional to the square root of the added modes. These non-BPS states will decay. The simplest decay process is a collision of a left moving excitation with a right moving excitation to give a closed string which leaves the brane. If the momenta are not exactly opposite, the outgoing string carries some momentum in the 9^{th} direction and we get a charged particle from the five dimensional point of view. Also note that the outgoing particle must be quantized in units of $1/R_9$, only particles that live on the brane can have fractional momenta. This means that outgoing particles have a very large mass and they are thermally supressed when R_9 is very small. All charged particles have masses of at least the compactification scale.

A careful calculation by [4, 5] shows that perturbative string theory calculation gives exactly the correct Hawking's black-body formula. It is even more striking that perturbation theory gives the correct grey body factors, which is just the energy-dependent absorbtion cross-section.

6 Discussion

We have looked at black holes on type II supergravity compactified to five dimensions. The extremal black holes have an exact correspondance with a superposition configuration of a collection of D-p-branes. The entropy was found not to depend on any continous parameters. We then viewed these same D-p-brane configurations from the point of view of perturbative string theory. Counting the microscopic states we found that the entropy excatly agrees with that found for the extremal black hole.

In the near extremal case, one can do the counting only when the string coupling is weak. To compare to the black hole limit, we need to make the coupling bigger, but string perturbation theory breaks down, so we might expect large corrections. However precise calculations show that the two answers agree. D-branes account for some non-perturbative effects, so it is hoped that they are accounting for all the neccessary ones for this counting. The results certainly seem to imply that.

Hawking radiation is viewed as the collision of a left moving and a right moving excitation on the brane resulting in a decay into a closed string which leaves the brane. The Hawking temperature is precisely the classical result.

If this picture is at least qualitatively correct than there would be no information loss, since all the information would be encoded in the open strings that live on the brane which sits at the horizon (for an extremal black hole). So the classical region which exists inside the horizon would be an effective description of these open strings, or vice versa. There should be a way to describe what happens when an observer falls into the black hole in terms of some closed strings that fall through the horizon. The information loss could be analyzed in the near-extremal case, but our lack of control on the strong coupling means that we cannot say anything definite about it.

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