LQG, Strings and all that

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1 Introduction

In this paper we hope to give an introduction to loop quantum gravity (LQG) ideas as well as to discuss a recent attempt to use LQG methods to quantize the free Nambu-Goto string.

Probably the main reason loop quantum gravity is interesting is because it provides a consistent *background independent* method of quantizing gravity. We want to stress background independence here, because in the other potential candidate of quantum gravity theory – string theory, there is no background independent formulation (yet). Exactly why is it so important, and what does it actually mean?

Well the way background *dependent* quantization is done is the following. One chooses some background metric (which can be Minkowski metric, or AdS, or something else), and then one tries to understand the excitations on that background (gravitons) and their interaction properties. However that seems to violate the main ideas of general relativity (GR). *There is no background metric in general relativity!* Background dependent formulation makes sense only when the excitations are small, and so it has a semiclassical point of view rather than a quantum one. Loop quantum gravity is a *non-perturbative* attempt to quantize gravity, which is actually background independent.

In section 2 of this paper we will give a short review of classical ADM and connection formalisms of GR, and introduce the loop quantization ideas in the sections 3, 4. Finally in section 5 we will discuss the recent attempt [15] of applying LQG methods to quantize Nambu-Goto string and mention an even more recent result of impossibility to algebraically quantize the Nambu-Goto string.

2 ADM and Connection Formalisms

We begin here by giving a short introduction to Arnowitt-Deser-Misner (ADM) formulation of GR, and then use it to introduce the gauge theory formulation.

The ADM formulation is a Hamiltonian or canonical formulation of GR. The canonical scheme is based on a phase space Γ. The phase space is a covariant notion, being simply the space of solutions of the equations of motion (modulo gauges). However in order to coordinatize Γ explicitly we will break the explicit covariance and split the 4-d spacetime $\mathcal{M}$ into 3-d space plus “time”, i.e. we take $\mathcal{M}$ to be diffeomorphic to $\mathbb{R} \times M$. This split of course does not correspond to the most general case, but it is also not very restrictive since roughly any manifold with a smooth metric with everywhere Lorentzian signature is necessarily of that topology [4]. The indices $\mu, \nu = 0, 1, 2, 3$ will label spacetime coordinates and $a, b = 1, 2, 3$ space coordinates on M.

The basic variables on phase space are taken to be the intrinsic 3-metric $q_{ab}(x)$ on M and the extrinsic curvature K_{ab} of M. The 3-metric and the extrinsic curvature are the spatial components of the following tensors

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \ K_{\mu\nu} = q_{\mu\rho} q_{\nu\sigma} \nabla^\rho n^\nu, \tag{1}$$

where n^ν is the vector field normal to M. Notice that both the intrinsic metric and the extrinsic curvature are spatial, that is, their contraction with n^ν vanishes, which explains why all of their information is contained in the spatial components, e.g. $q_{ab} = q_{\mu\nu} S^\mu_a S^\nu_b$. Here S^μ_a is the tangent to M vector field.

One can then proceed and define a Hamiltonian in a way it's usually done for singular Lagrangians [9], and get a classical Hamiltonian formulation of GR. We will not need that, so we refer the interested reader to [16]. Note however, that since the Lagrangian is singular the system is constrained. It can be shown that there are four constraints (one for each spacetime diffeomorphism), the explicit form of which we won't need. We will denote them by C and V_a and refer to as *Hamiltonian and Spatial Diffeomorphism constraints* respectively.

We now go on and construct the gauge theory formulation introduced first by Ashtekar [1]. Expressing the 3-metric in terms of (local) triads $e_a^i(x)$

$$q_{ab} = e_a^i e_b^i, \tag{2}$$

where $i = 1, 2, 3$. This introduces additional $SU(2)$ gauge symmetry into the theory, which geometrically corresponds to arbitrary local frame rotations.

Defining the spin connection Γ_a^i through the equation

$$\partial_a e_b^i - \Gamma_{ab}^c e_c^i + \epsilon_{ijk} \Gamma_a^j e_b^k = 0, \tag{3}$$

where Γ_{ab}^c are the Christoffel symbols associated with q_{ab}. We then define the *connection*

$$A_a^i = \Gamma_a^i + \beta K_{ab} e_i^b, \quad E_i^a = \frac{1}{\beta}\sqrt{\det(q)}\, e_i^a, \tag{4}$$

where β is called the *Immirzi* parameter and is a non-zero complex number. We will however use real and positive β throughout this paper[1]. Finally we introduce the $SU(2)$ *Gauss constraint*

$$G_i = \partial_a E_i^a + \epsilon_{ijk} A_a^j E_k^a, \tag{5}$$

with ϵ_{ijk} the structure constants of $SU(2)$, which we would encounter in the canonical formulation of any $SU(2)$ gauge theory.

Taking A_a^i to be the canonical variables, the E_i^a will be the canonically conjugate momenta. A theorem then states [1] that if one takes the phase space coordinatized by (A_a^i, E_i^a) with Poisson brackets

$$\{E_i^a(x), E_j^b(y)\} = \{A_a^i(x), A_b^j(y)\} = 0, \quad \{A_a^i(x), E_j^b(y)\} = \kappa \delta_b^a \delta_j^i \delta(x-y), \tag{6}$$

and constraints G_i, C, V_a, solves the constraint $G_i = 0$ and determines the Dirac observables with respect to it, one gets back the original ADM phase space with constraints C, V_a.

At this point it doesn't seem like all this can help. We made things even more complicated introducing new degrees of freedom and corresponding constraints. However this extended phase space allows us to *formulate canonical GR in the language of a canonical gauge theory* where A plays the role of an $SU(2)$ connection with canonically conjugate electric field E. It is precisely this that allows to actually quantize the theory, since we can make use of all the powerful techniques developed for canonical quantization of gauge theories.

3 Loop Quantization

3.1 Loop Algebra

On M we define a smooth, Lie algebra valued connection 1-form A

$$A(x) = A_a^i(x) \tau_i dx^a \equiv A_a(x) dx^a, \tag{7}$$

[1] The use of the set of canonical variables involving a complex connection leads to a simplification of the Hamiltonian constraint. With the use of a real connection, the constraint loses its simple polynomial form. Initially this was considered as a serious obstacle for quantization. However, Thiemann succeeded in constructing a Lorentzian quantum Hamiltonian constraint [13] in spite of the non-polynomiality of the classical expression.

where x are local coordinates on M, $A_a^i(x) \in C^\infty(M)$, and $\tau_i = \frac{i}{2}\sigma_i$ are the $SU(2)$ generators in the fundamental representation, and σ_i are the Pauli matrices. We also define the corresponding "electric field" vector field

$$E(x) = 2E_i^a(x)\tau_i\partial x_a \equiv E^a(x)\partial x_a \tag{8}$$

Certain classical quantities play a very important role in the quantum theory. In particular they will form the algebra of the "elementary" operators, i.e. we will promote Poisson brackets to commutators for them (without adding central charges). This should be viewed in the same way as x and $i\hbar d/dx$ in quantum mechanics.

These important quantities are the trace of the holonomy of the connection, which is labelled by loops on the three manifold M, and the higher order loop variables, obtained by inserting the E field (in n distinct points) into the holonomy trace. To be more precise let us define the holonomy of A to be the parallel propagator of A_a along the loop α

$$U_\alpha(s_1, s_2) = \mathcal{P}exp\left\{\int_{s_1}^{s_2} A_a(\alpha(s))ds\right\}, \tag{9}$$

where $\mathcal{P}$ denotes path ordering. The holonomy is the unique solution to the ODE

$$\frac{d}{ds}U_\alpha(1,s) = \frac{d\alpha_a(s)}{ds}A_a(\alpha(s))U_\alpha(1,s), \;\; U_\alpha(1,1) = \mathbb{1}. \tag{10}$$

The aforementioned quantities can now be defined as

$$\mathcal{T}[\alpha] = -\mathrm{Tr}[U_\alpha] \tag{11a}$$

$$\mathcal{T}^a[\alpha](s) = -\mathrm{Tr}[U_\alpha(s,s)E^a(s)] \tag{11b}$$

$$\mathcal{T}^{a_1a_2}[\alpha](s_1,s_2) = -\mathrm{Tr}[U_\alpha(s_1,s_2)E^{a_2}(s_2)U_\alpha(s_2,s_1)E^{a_1}(s_1)] \tag{11c}$$

$$\mathcal{T}^{a_1\ldots a_n}[\alpha](s_1,\ldots,s_n) = -\mathrm{Tr}[U_\alpha(s_1,s_n)E^{a_n}(s_n)U_\alpha(s_n,s_{n-1})\ldots E^{a_1}(s_1)], \tag{11d}$$

where the points $s_1, s_2, \ldots, s_n \in \alpha$.

The loop observables coordinatize the phase space and have a closed Poisson algebra, denoted the *loop algebra*. The loop algebra has an interesting geometrical property. For instance, the Poisson bracket between $\mathcal{T}[\alpha]$ and $\mathcal{T}^a[\beta](s)$ is non vanishing only if $\beta(s)$ lies over α. More precisely

$$\{\mathcal{T}[\alpha], \mathcal{T}^a[\beta](s)\} = \Delta^a[\alpha, \beta(s)]\left[\mathcal{T}[\alpha\#\beta] - \mathcal{T}[\alpha\#\beta^{-1}]\right], \tag{12}$$

where

$$\Delta^a[\alpha, x] = \int ds\frac{d\alpha^a(s)}{ds}\delta^3(\alpha(s) - x) \tag{13}$$

is a vector distribution with support on α and $\alpha\#\beta$ is the loop obtained starting at the intersection between α and β, and following first α and then β. β^{-1} is β with reversed orientation.

An alternative to taking the full loop observables (11) as elementary variables is to take only the holonomies $\mathcal{T}[\alpha]$ and the conjugate momenta $E_i^a(x) \sim \frac{\delta}{\delta A_a^i(x)}$. We defer the discussion of this to section 3.4.

3.2 Loop Quantum Gravity (LQG)

As already discussed in the previous section, to define the quantum theory one has to make a choice of "elementary" operators (such as x and p in quantum mechanics, or creation and annihilation operators) on a Hilbert space $\mathcal{H}$, which will be the space of quantum states. To assure that the quantum Heisenberg equations have the correct classical limit, the quantum algebra of the elementary operators has to be isomorphic to the classical Poisson algebra of the elementary operators. This procedure is basically given by promoting Poisson brackets to commutators. In loop quantum gravity the algebra of elementary observables is chosen to be the loop algebra, defined above. The main reason for choosing these variables is because they are background independent (also because it actually works), however this choice is not harmless and the ultimate answer whether or not this approach is right will probably be given by experiment (at least at our current level of understanding classical to quantum theory transition).

Let us now proceed and define quantum states on our Hilbert space $\mathcal{H}$. We will do that by defining functionals $\Psi(A)$ of the smooth connection. These functionals will form a linear space, which we will promote to a Hilbert space by defining an inner product.

Define a graph $\Gamma_n = \{\gamma_1, \ldots, \gamma_n\}$ as a finite collection of n oriented piecewise smooth curves (edges) γ_i, $i = 1, \ldots, n$, embedded in the 3-manifold M, that meet, if at all, only at their endpoints. Let $U_i(A) \equiv U_{\gamma_i}$, $i = 1, \ldots, n$ be the holonomy of the connection A along γ_i. $U_i(A)$ is an element of $SU(2)$. Pick a function $f(g_1, \ldots, g_n)$ on $[SU(2)]^n$

$$f : [SU(2)]^n \to \mathbb{C}. \tag{14}$$

The graph Γ and the function f determine a functional of the connection as follows

$$\Psi_{\Gamma,f}(A) = f(U_1(A), \ldots, U_n(A)). \tag{15}$$

These states are called *cylindrical states or functions*. An important property of the cylindrical states which is essential in the definition of the scalar product is the following. If Γ is a subgraph of Γ' we can always write

$$\Psi_{\Gamma,f}(A) = \Psi_{\Gamma',f'}(A), \tag{16}$$

by choosing f' independent from the U_i's of the edges that are in Γ' but not in Γ. Thus, given any two cylindrical functions we can always view them as having the same graph, by just taking the union of the two graphs. Given this property we can now define a scalar product for any two cylindrical functions by

$$\langle \Psi_{\Gamma,f} | \Psi_{\Gamma,g} \rangle = \int_{[SU(2)]^n} dU_1 \ldots dU_n \overline{f(U_1, \ldots, U_n)} g(U_1, \ldots, U_n), \tag{17}$$

where dU_i is the Haar measure on $SU(2)$. This scalar product extends by linearity to finite linear combinations of cylindrical functions and defines a well-defined scalar product on the space of these linear combinations. Completing the space of these linear combinations in the Hilbert norm ($\|\Psi_{\Gamma,f}\| = \langle \Psi_{\Gamma,f} | \Psi_{\Gamma,f} \rangle^{1/2}$) we get the unconstrained quantum state space of loop gravity. An important property of the scalar product (17) is that it is invariant under both the diffeomorphism group and the group of the local $SU(2)$ transformations $A \to \phi^{-1}A$, $\phi : M \to M$ and $A \to A_\lambda = \lambda^{-1}A\lambda + \lambda^{-1}d\lambda$, $\lambda : M \to SU(2)$ [7].

We will use Dirac notation and write

$$\Psi(A) = \langle A | \Psi \rangle\,, \tag{18}$$

similar to $\psi(x) = \langle x | \psi \rangle$ in quantum mechanics. As in that case $|A\rangle$ is not a normalizable state.

3.3 Spin Network States

We now construct an orthonormal basis in the Hilbert space $\mathcal{H}$, denoted *spin network basis*.

Before defining the spin network basis, we introduce the "old" *loop state basis*. Given a loop α in M, there is a normalized state $\Psi_\alpha(A)$ in $\mathcal{H}$, which is obtained by taking $\Gamma = \alpha$ and $f(g) = -\mathrm{Tr}(g)$, i.e.

$$\Psi_\alpha(A) = -\mathrm{Tr}U_\alpha(A) \equiv \langle A | \alpha \rangle\,. \tag{19}$$

These states are called *loop states*. They are normalizable, and products of loop states are normalizable as well. We denote by α also a multiloop, i.e. a collection of possibly overlapping loops $\{\alpha_1, \ldots, \alpha_n\}$ and call

$$\Psi_\alpha(A) = \Psi_{\alpha_1}(A) \times \ldots \times \Psi_{\alpha_n}(A) \tag{20}$$

a multiloop state.

Before the introduction of spin network states, multiloop states were the main tool in loop quantum gravity, however although they span the Hilbert space $\mathcal{H}$, they are linearly dependent, which is not the case for spin networks, which form a true basis. Let us now go on to spin networks.

Consider a graph Γ with n edges γ_i, $i = 1, \ldots, n$. To each edge γ_i we assign a non-trivial irreducible representation of $SU(2)$ which is labelled by its spin j_i. The integer $2j_i$ is called the *color* of the edge. The Hilbert space on which this irreducible spin-j_i representation is defined is denoted by $\mathcal{H}_{j_i}$.

Consider next a node p, with valency k, i.e. with k edges $\gamma_1, \ldots, \gamma_k$ meeting at that node. Let the edges be colored by $j_1, \ldots, j_k$. The corresponding Hilbert spaces are then $\mathcal{H}_{j_1}, \ldots, \mathcal{H}_{j_k}$. Denote the tensor product of these spaces by

$$\mathcal{H}_p = \mathcal{H}_{j_1} \otimes \ldots \otimes \mathcal{H}_{j_k}. \tag{21}$$

Now fix an orthonormal basis in $\mathcal{H}_p$. A choice of an element N_p of the basis is then called the *coloring of the node* p.

A non-gauge invariant *spin network* S is then defined as a colored graph in which both edges and nodes are colored. A spin network is thus a triple $S = (\Gamma, \{j_i\}, \{N_i\})$.

Now we can define a *spin network state* $\Psi_S(A)$ as

$$\Psi_S(A) \equiv \langle A|S\rangle = f_S(U_1(A), \ldots, U_n(A)). \tag{22}$$

The function f_S is constructed by taking the holonomy along each edge in the corresponding to the edge $SU(2)$ irrep and contracting with the color vectors N_p of the nodes

$$\Psi_S(A) = f_S(U_1(A), \ldots, U_n(A)) = \otimes_{\text{edges } i\in\Gamma} R^{j_i}(U_i) \cdot \otimes_{\text{nodes } p\in\Gamma} N_p. \tag{23}$$

Here $R^{j_i}(U_i)$ is the representation matrix of the holonomy U_i in the spin-j_i irrep of $SU(2)$.

By varying the graph, the colors of the edges and of the nodes, we get a collection of states which turn out to be normalized. Finally using Peter-Weyl theorem on can show that any two distinct states Ψ_S are orthonormal

$$\langle \Psi_S|\Psi_{S'}\rangle = \delta_{S,S'} \equiv \delta_{\Gamma,\Gamma'}\delta_{\vec{j},\vec{j}'}\delta_{\vec{N},\vec{N}'}, \tag{24}$$

and that if Ψ is orthogonal to every spin network state, then $\Psi = 0$.

To finalize the construction of the spin network states we will project onto $SU(2)$ gauge invariant states. The resulting Hilbert space of gauge invariant states will be denoted by $\mathcal{H}_0 \subset \mathcal{H}$.

An $SU(2)$ gauge transformation acts on a spin network state simply by $SU(2)$ transforming the coloring of the nodes N_p. Then to find the complete set of gauge invariant states, we decompose the Hilbert space $\mathcal{H}_p$, which is the tensor product of irreps meeting at node p, into irreducible parts

$$\mathcal{H}_{j_1} \otimes \ldots \otimes \mathcal{H}_{j_k} = \oplus_J (\mathcal{H}_J)^{k_J}, \tag{25}$$

where k_J denotes the multiplicity of the spin-J irrep. Keeping then only the singlet, i.e. $J = 0$ subspace, and choosing an arbitrary basis in this subspace and assigning color N_p from that basis to the node p gives the so called *gauge invariant spin network*. The N_p's are also called *intertwiners*.

Consider some examples. If there is only one edge connected to the node (it's 1-valent), then since the color of the edges cannot be zero, there is no singlet. If the node is 2-valent, then to have a singlet in $j_1 \otimes j_2$ we need $j_1 = j_2$ and so $(N_p)_{j_1 j_2} = \delta_{j_1 j_2}$. For the 3-valent state the invariant tensor N_p is proportional to the Wigner $3j$-coefficient (i.e. the color of each edge is not larger than the sum of the other two, and the sum of the three colors is even) and so on. Thus the way a gauge invariant spin network is constructed is purely algebraic.

We summarize the main properties of the (gauge invariant) spin network states:

- Spin network states are normalizable.
- They are $SU(2)$ gauge invariant.

- Each spin network state can be decomposed into a finite linear combination of products of loop states.
- They form an orthonormal basis for $\mathcal{H}_0$, the Hilbert space of gauge invariant states.
- The scalar product between two spin network states can be computed graphically and algebraically [6].

3.4 Operators

Since we now have the kinematical Hilbert space (kinematical means that we did not impose the diffeomorphism and Hamiltonian constraints yet), we can define the quantum operators, corresponding to the loop algebra variables (11). The operator $\mathcal{T}[\alpha]$ for instance will acts simply by

$$\hat{\mathcal{T}}[\alpha]\Psi(A) = -\mathrm{Tr}U_\alpha(A)\Psi(A). \tag{26}$$

In diagrammatic notation it simply adds a loop to multiloops

$$\hat{\mathcal{T}}[\alpha]\left|\Psi\right\rangle = \left|\alpha\right\rangle\left|\Psi\right\rangle. \tag{27}$$

Instead of looking at the action of the other loop algebra variables, we come back to a point mentioned in section 3.1. Recall that the conjugate momentum for the connection was $E^a_i(x)$ which in quantum theory will be promoted to the operator

$$E^a_i(x) \sim -i\hbar\frac{\delta}{\delta A^i_a(x)}. \tag{28}$$

However this object is not an operator, but rather an operator valued distribution, and so it has to be smeared, i.e. integrated over suitable test functions. On the 3-manifold M vector density E^a is actually dual to a 2-form $E = \epsilon_{abc}E^a dx^a dx^c$, so the natural thing to do is to integrate it over a 2-surface. This gives a well-defined and finite operator

$$\hat{E}^i(\Sigma) = -i\int_\Sigma d\sigma^1 d\sigma^2 n_a(\overrightarrow{\sigma})\frac{\delta}{\delta A^i_a(x(\overrightarrow{\sigma}))}, \tag{29}$$

where

$$n_a(\overrightarrow{\sigma}) = \epsilon_{abc}\frac{\partial x^b(\overrightarrow{\sigma})}{\partial\sigma^1}\frac{\partial x^c(\overrightarrow{\sigma})}{\partial\sigma^2} \tag{30}$$

and ϵ_{abc} is the Levi-Civita tensor.

One can then go on and construct the action of this operator on the Hilbert space states. We will just quote the results here, for a detailed discussion see [7]. The gauge-invariant operator one can construct out of the operator (29) is

$$\hat{E}^2(\Sigma) = \hat{E}^i(\Sigma)\hat{E}^j(\Sigma). \tag{31}$$

The action of this operator on a spin network which intersects Σ only at one point is

$$\hat{E}^2(\Sigma)\left|S\right\rangle = j(j+1)\left|S\right\rangle, \tag{32}$$

where j is the spin associated with the edge that intersects Σ. One can generalize this operator taking appropriate limit so that it behaves properly [7], to give an operator $\mathcal{U}(\Sigma)$, which (roughly) acts on a spin network by[2]

$$\mathcal{U}(\Sigma)\left|S\right\rangle = \sum_i\sqrt{j_i(j_i+1)}\left|S\right\rangle, \tag{33}$$

[2]The spectrum of the operator is a little different if the spin network has a node on the surface Σ.

where the sum is over the edges that intersect the surface Σ.

This operator may not seem so exciting at first, but it turns out that the classical limit of this operator is simply the surface area of the surface Σ! So the first non-trivial result one gets in LQG is that the area is actually quantized. One can also show that the volume operator will have quantized spectrum. Note however that for some reason there isn't an analogous statement for the length. Either it doesn't make sense to ask about length in quantum theory, or it is just a technicality which haven't been solved yet.

3.5 Diffeomorphism Constraint

We now go on and try to impose the diffeomorphism constraint, which simply means that we have to somehow project diffeomorphism invariant states out of our gauge-invariant Hilbert space $\mathcal{H}_0$. A rigorous construction is done in [2]. The result is that the diffeomorphism invariant objects are not graphs embedded in a 3-manifold M, but rather abstract graphs specified by giving the edges, the nodes and their mutual relationship (plus the colors). This is simply because a diffeomorphism transformation can displace the nodes and edges in any way it likes, but can't change the topology of the graph. The equivalence class of diffeomorphism related spin network states is called a *knot state* and is denoted by $|s\rangle$, and one can show that these form a basis in the Hilbert space $\mathcal{H}_{diff} = \mathcal{H}_0/Diff(M)$. The basis is denoted *knot state basis*.

We have seen above that each edge of the graph can be seen as carrying a quantum of area. One can also show that each node in a way carries a quantum of volume. Thus an s-knot can be seen as an elementary quantum excitation of space formed by "chunks" of space (nodes) separated by sheets of surface (edges) with quantized area. The key point here is that an s-knot does not live on a manifold it itself defines the manifold of quantum space-time (actually at this point just space). Put in a different way, the s-knot is not "somewhere" in space, it itself defines the "where".

3.6 Hamiltonian Constraint

We come to the final constraint left – the Hamiltonian constraint. We should stress that we are not going to solve it explicitly, and that is not surprising, if we did we would solve completely the system.

Again we will not go into details here but quote the results. First of all one can actually construct a quantum operator $\hat{H}$ corresponding to the Hamiltonian constraint, which is the first good news. However, calculating the action of the constraint on spin network states, the result turns out to be not equal to zero, meaning that they are not invariant. In fact $\hat{H}$ acts only on the nodes of the network, adding an edge

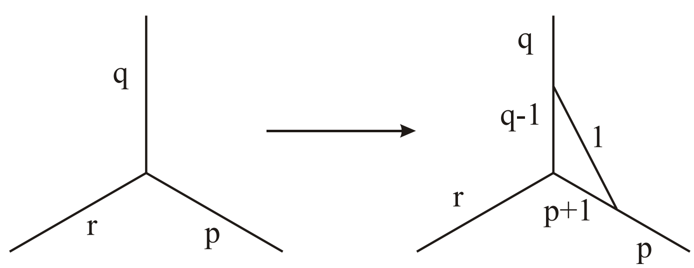

Figure 1: Hamiltonian constraint action on a trivalent node.

There is a very tempting physical interpretation of this. If one defines the physical scalar product by

$$\langle s|s'\rangle_{phys} = \langle s|P|s'\rangle\,, \tag{34}$$

where P is the Hamiltonian constraint projection operator (see e.g. [7] for definition), then expanding out the projection operator one gets a very interesting picture. In the zeroth order one just gets $\langle s|s'\rangle_{phys} =$

$\langle s|s'\rangle$, and so it will vanish unless the networks are topologically the same, however the first order term has the Hamiltonian constraint operator $\langle s|\hat{H}|s'\rangle$, summed over all possible ways it can act on the network. This can be interpreted as a Feynman diagram for "time propagation" of networks! The Hamiltonian constraint actually gives a usual QFT-like picture for loop quantum gravity. The elementary vertex given by the action of the Hamiltonian constraint above is the elementary interaction of the theory.

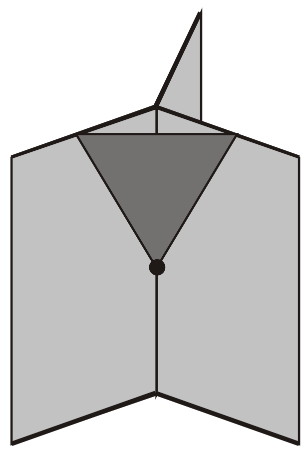

Figure 2: The elementary vertex.

This idea of viewing the theory in a Feynman diagram like way and further extensions of it go under the name of *spin foams*.

4 Physical Results of Loop Quantum Gravity

We summarize here the results we got, and add some other interesting results.

- The space seems to be discrete at the Planck scale. The discussion above doesn't necessarily prove that the physically measurable space will be quantized, since for instance the area operator we defined is not diffeomorphism invariant. However one can argue (under some additional hypothesis) that non-diff-invariant observables in the pure gravity theory correspond to diff-invariant observables in matter+gravity theory (one can imagine gauge-fixing the matter variables and using matter location as coordinates). Of course there is no way of measuring/probing that discreteness at the moment, since there is no experimental access to Planck scale physics.

- There are independent ways of calculating black hole entropy in LQG and they give the Bekenstein-Hawking entropy up to a constant. More precisely in LQG

$$S = \frac{c}{\beta}\frac{k}{4\hbar G_{Newton}}A, \tag{35}$$

  where β is the Immirzi parameter defined above, and $c \sim 1/4\pi$ is a constant. So if we take $\beta = c$, then we will recover the usual Bekenstein-Hawking entropy. At the moment the precise significance of the condition $\beta = c$ is not quite clear.

- Surprisingly there are exact solutions of the Hamiltonian constraint, e.g. the Kodama state [12] describing a quantum deSitter space, which also has gravitons moving on the classical deSitter spacetime background, in the long wavelength limit.

Of course the theory has a lot of open questions as well, in particular the classical limit is not very well investigated, also there is no general technique for extracting physical information similar to the S-matrix in QFT. In particular if one could calculate scattering amplitudes then one could compare the LQG with string theory predictions.

5 LQG-String

Recently there appeared an interesting article [15], which claims to give a consistent quantization of the Nambu-Goto string using LQG methods, but which has properties that differ drastically from the usual quantization of the string [10]. In particular the author does not find a critical dimension and the mass spectrum is an arbitrary non-negative spectrum, thus in particular the tachyon is gone (as well as the rest of the string spectrum).

Before we present the "LQG-String" we give an introduction to the so called *Pohlmeyer string*.

5.1 Nambu-Goto String and its Constraints

We start of with the bosonic p-brane with $p \leq 2$. The action, which is the analog of Nambu-Goto action for the string, is simply the "area" of the brane

$$S[X] = \frac{1}{\alpha'} \int_M d^p\sigma \sqrt{-\det(h_{ab})}, \tag{36}$$

where h_{ab} is the induced metric on the brane and is equal to

$$h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu, \tag{37}$$

$\alpha, \beta = 0, \ldots, p-1$. X^μ's are the target space coordinates which is taken to have Minkowski metric and to have some dimension D.

We then proceed to the canonical analysis. We assume that $M = \mathbb{R} \times \sigma$ where σ is a $(p-1)$-dimensional manifold and thus split the coordinates σ into (t, x) accordingly. The momentum conjugate to X^μ is then given by

$$\pi_\mu = \frac{\delta S}{\delta \dot{X}^\mu} = -\sqrt{-h} h^{t\alpha} \partial_\alpha X_\mu, \tag{38}$$

where $h \equiv \det(h_{\alpha\beta})$, and h with upper indices is the inverse of $h_{\alpha\beta}$.

We also introduce Poisson brackets, which by definition give

$$\{X^\mu(t,x), X^\nu(t,x')\} = \{\pi_\mu(t,x), \pi_\nu(t,x')\} = 0, \ \ \{X^\mu(t,x), \pi_\nu(t,x')\} = \delta^\nu_\mu \delta(x - x'). \tag{39}$$

The Lagrangian (36) is singular, therefore we should have some constraints, and as one can see directly from (38)

$$\begin{aligned} D_a &\equiv \pi_\mu \partial^\mu_a \approx 0 \\ C &\equiv \frac{1}{2}[\pi_\mu \pi^\mu + \det(q)] \approx 0, \end{aligned} \tag{40}$$

where

$$q_{ab} = h_{ab}, \tag{41}$$

$a, b = 1, \ldots, p-1$, and symbol ≈ 0 should be read "weakly zero" and means that the expressions vanish but may have non-vanishing Poisson brackets with some canonical variables (for a very good introduction to canonical treatment of constrained systems see [9]). These are the analogs of the diffeomorphism and Hamiltonian constraints in LQG, and so we will call D_a and C diffeomorphism and Hamiltonian constraints respectively.

As one can easily check these constraints are in fact first class (i.e. the Poisson brackets among themselves vanish weakly), so that there are no secondary constraints. One can in fact compute the Poisson brackets for the smeared versions of the constraints to be

$$\begin{aligned} &\{D(\overrightarrow{N}), D(\overrightarrow{N}')\} = D(\mathcal{L}_{\overrightarrow{N}}\overrightarrow{N}') \\ &\{D(\overrightarrow{N}), C(N')\} = C(\mathcal{L}_{\overrightarrow{N}}N') \\ &\{C(N), C(N')\} = \int_\sigma d^{p-1}x(\partial_a N'N - \partial_a NN')[det(q)q^{ab}]D_b, \end{aligned} \tag{42}$$

where $\mathcal{L}$ denotes the Lie derivative, and we have smeared the constraints with smooth test functions (of rapid decrease), i.e. $C(N) = \int_\sigma d^{p-1}xNC$ and $D(\overrightarrow{N}) = \int_\sigma d^{p-1}xN^a D_a$.

We notice here that precisely when $p = 2$, $det(q)q^{ab} = 1$. Thus the string is singled out from other p-branes by the fact that its constraints from an honest Lie Poisson algebra. Notice that in the discussion above we did not introduce a worldsheet metric and thus the only local worldsheet symmetry is Diff(M).

Henceforth we will consider only the closed bosonic string – $p = 2$ and $M = \mathbb{R} \times S^1$.

5.2 Pohlmeyer String

We briefly sketch here the construction of the *Pohlmeyer Charges* which form a basis for all gauge-invariant observables of the Nambu-Goto string (thus they are the analog of spin networks).

The idea is to use the method of *Lax pairs*. One reformulates the equations of motion so that they become of the form $\partial_\alpha M = [A_\alpha, M]$, where $M(t,x)$ is an $N \times N$ matrix and A_α is a "connection", i.e. a 1-form on $\mathbb{R} \times S^1$. The integrability condition for such a Lax pair M, A is the zero curvature condition $F_{\alpha\beta} = \partial_{[\alpha}A_{\beta]} + [A_\alpha, A_\beta] = 0$ and one looks for M,A in such a way that $F_{\alpha\beta} = 0$ is equivalent to the equations of motion. Given such a setup it follows that the functions $\mathrm{Tr}(M^n)$, $n = 1, \ldots, N$ are constants of motion.

The "Hamiltonian" for the string is given by an arbitrary linear combination of the constraints

$$H(u, v) = C(u) + D(v), \tag{43}$$

where u and v are arbitrary test functions. We introduce new variables to use instead of the canonical ones, which are just the momenta of left and right movers

$$Y^\mu_\pm = \pi^\mu \pm X^{\mu\prime}, \tag{44}$$

where prime denotes differentiation with respect to x.

"Time" evolution is then defined by

$$\dot{Y}_\pm = \{H(u, v), Y^\mu_\pm\} = [(v \pm u)Y^\mu_\pm]'. \tag{45}$$

The invariants that we will construct will be functions of $Y^\mu_\pm$, and so we will have to construct them in such way that they will not depend on u and v.

Let τ_I be a basis of $GL(n, \mathbb{R})$ and T^I_μ some complex valued matrices. We define then the following connection

$$A^\pm_x = Y^\mu_\pm T^I_\mu \tau_I \equiv Y^\mu_\pm T_\mu, \;\; A^\pm_t = (v \pm u)A^\pm_x. \tag{46}$$

The zero curvature condition then mimics the equations of motion

$$F^\pm_{tx} = \partial_t A^\pm_x - \partial_x A^\pm_t + [A^\pm_t, A^\pm_x] = \partial_t A^\pm_x - \partial_x A^\pm_t = \partial_t A^\pm_x - \{H(u, v), A^\pm_x\} = 0. \tag{47}$$

Now we need to find a suitable matrix to form a Lax pair. It turns out that the right choice is actually the holonomy of the connection

$$h_c(A^\pm) = \mathcal{P}exp\left(\int_c ds \frac{dc^\alpha}{ds} A^\pm_\alpha\right). \tag{48}$$

If we now choose our path to be $c_{t,x} = t \times [x, x + 2\pi)$ (recall that our worldsheet is $\mathbb{R} \times S^1$) and denote $h_{t,x}(A^\pm) \equiv h_{c_{t,x}}(A^\pm)$, then it can be easily shown [15] that the zero curvature condition is equivalent to

$$\partial_\alpha h_{t,x}(A^\pm) = [h_{t,x}(A^\pm), A^\pm_\alpha(t,x)]. \tag{49}$$

So we claim then that $\forall n \leq N$ the functions

$$T^n_{t,x}(A^\pm) = \mathrm{Tr}_N([h_{t,x}(A^\pm]^n) \tag{50}$$

are independent of both x and t, which follows immediately from (49) and the cyclicity of the trace. Even more their Poisson brackets with the Hamiltonian $H(u,v)$ vanish for all u and v (this follows from $\{H(u,v), h_{t,x}(A^\pm)\} = \partial_t h_{t,x}(A^\pm)$).

We can trade the power $n \leq N$ for considering increasing rank N of the matrices τ_I, and so the only interesting invariant is

$$Z^T_\pm = T^1(A^\pm) = N + \sum_{n=1}^{\infty} Z^{\mu_1 \ldots \mu_N}_\pm \frac{\mathrm{T}_N(T_{\mu_1} \ldots T_{\mu_N})}{N}. \tag{51}$$

Since the matrices T^μ are arbitrary we conclude that the expansion coefficients themselves are invariants. These are the *Pohlmeyer Charges*

$$\begin{aligned} Z^{\mu_1 \ldots \mu_N}_\pm &= [R^{\mu_1 \ldots \mu_N}(x) + R^{\mu_2 \ldots \mu_N \mu_1}(x) + \ldots + R^{\mu_N \mu_1 \ldots \mu_{N-1}}(x)] \\ R^{\mu_1 \ldots \mu_N}_\pm(x) &= \int_x^{x+2\pi} dx_1 \int_{x_1}^{x+2\pi} dx_2 \ldots \int_{x_N-1}^{x+2\pi} dx_N Y^{\mu_1}_\pm(x_1) \ldots Y^{\mu_N}_\pm(x_N). \end{aligned} \tag{52}$$

The *Pohlmeyer Charges* form a complicated, closed subalgebra of the full Poisson algebra of the string. One can show that together with the Lorentz generators they provide a *complete* system of invariants for the string in the sense that one can reconstruct $X^\mu(t,x)$ up to gauge transformations and up to translations in the direction of p^μ.

These invariants are not linearly independent but one can also construct linearly independent, so called *standard invariants* (see [15] for a simplified discussion).

Pohlmeyer's idea of quantizing the string is roughly to first solve the constraint algebra and then to quantize it. We will follow however the usual pattern of quantizing first and getting the kinematical Hilbert space $\mathcal{H}$, then imposing the constraints to get the physical one $\mathcal{H}_{phys}$.

We should mention here a recent result about algebraic quantization of the Nambu-Goto string [3], which shows that if one tries to impose the usual quantization of the oscillators in the Fock space of the string and computes the resulting commutators for the Pohlmeyer charges, one encounters anomalies in the algebra. These anomalies arise in a similar manner to anomalies of the Lorentz generators in the light-cone gauge, which are removed by the right choice of the critical dimension and the ordering constant, however these anomalies cannot be removed in any way. Basically this means that in the usual quantization of the string (as well as in any other Fourier mode parametrization), *correspondence principle does not hold for the invariant charges of the theory*, which are supposed to be the basic observables of the system.

5.3 Quantization

We will not go into the details of the quantization here but only present the main ideas here, because that would require the knowledge of elements of Algebraic Quantum Field Theory (AQFT) which we do not expect from the reader. However everything is presented in great detail (as well as a short introduction to AQFT) in the original paper [15].

The main steps of quantization as usual are

- Choose the classical Poisson algebra that will be promoted to a quantum algebra of observables. This is again similar to choosing x and p in the usual quantum mechanics to have Poisson brackets replaced by commutators. As in the case of LQG we note here that this choice may or may not be as harmless as it may seem. To avoid boundedness issues one usually defines formally the commutation relations between the elementary operators and then considers their exponentials, e.g. $W = exp(itp)$ in QM.

- The next step is to find a suitable basis in the kinematical Hilbert space $\mathcal{H}$ and to construct the action of the operators on the states, again based on the elementary ones.

- The hardest step is then usually the process of imposing the quantum constraints.

- Finally one has to check that the quantum theory constructed makes any sense at all by checking that it has the correct classical theory in the long wavelength limit.

One can proceed with all these steps as is done in [15]. It seems that the main difference however in the quantization procedure from the usual one is the way one actually imposes the constraints. In the Old Covariant Quantization (OCQ) of the string, one imposes the constraints (weakly) via

$$\langle phys|\hat{C}|phys\rangle = 0, \tag{53}$$

for all physical states. Here $\hat{C}$ is the operator expression for the constraints (which is simply the energy-momentum tensor in usual notation).

However what the author does instead, using the GNS (Gel'fand-Naimark-Segal) construction, he finds a representation $\hat{U}_\phi$ of the classical constraint *group* elements ϕ, and requires invariance of physical state under the action of $\hat{U}_\phi$

$$\hat{U}_\phi\,|phys\rangle = |phys\rangle\,. \tag{54}$$

Clearly by construction there are no anomalies – the classical group is isomorphic to the quantum one, which gives no central charge[3], tachyon or critical dimension for the quantum string. In fact the author could reproduce *any non-negative mass spectrum*, which supported the quantum Pohlmeyer charges and even had the correct classical limit.

Although we skipped this point in section 3.5 when presenting loop quantization, but the way the diffeomorphism constraint is imposed in LQG is very similar to the construction above, while the Hamiltonian constraint is imposed in the usual way. It appears that LQG methods are not "quite canonical" from this toy-model of Nambu-Goto string.

6 Conclusion

We hope that we gave a satisfactory introduction to loop quantum gravity in this paper, as well as to Pohlmeyer charges.

It does not seem very plausible that the results of LQG-string [15] are actually true, since they have to much arbitrariness in them, and one would like to hope that there will be a uniqueness theorem, like Stone-von Neumann theorem of quantum mechanics, for the quantization of the string. We hope the answer to this question as well as any evidence for/against string theory and/or loop gravity will be found experimentally in our lifetime (which does not seem very probable considering the value of the Planck scale).

[3] One can wonder how is the no-go theorem by V. Kac circumvented, which states that $c = 0$ is possible only if one abandons positive-definite Hilbert space, or positive energy, or unitarity. Thiemann's construction is not however subject to the positive-energy condition, because the Virasoro algebra is not considered as a quantum field on some CFT, but just as the generators of the symmetry. For more details on this see [5].

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