Inflation and Supergravity No-Go Theorem

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Abstract

We introduce the motivation for inflation as an additional phase of the early universe and then discuss some of the ways in which its theoretical grounding is being approached from the point of view of stringy cosmology, and, specifically, its low-energy regime—supergravity. F-term and D-term models are discussed and some limits on the admissible potentials are derived.

A supergravity no-go theorem is then discussed. We prove that a very large class of ten-dimensional supergravity models does not have a stable compactification into either de Sitter or Minkowski space. We conclude that string-theoretic non-perturbative methods must be used to create such vacua and only then can inflation be properly modelled and we give an example for just such a calculation. Unfortunately, even though the moduli remain fixed, no inflation occurs. We conclude with a statement that no working inflationary model with stable moduli has yet been found.

1 Introduction

In the early eighties, a cosmology dominated by a scalar field with a positive stationary value for its potential was proposed to solve some of the outstanding problems of traditional Big-Bang cosmology. It was to be named inflation and has become part of the every cosmologist's toolkit, being used to resolve all kinds of issues: from the uncanny flatness of the universe to the matter density perturbations (see §2.1). Unfortunately, there is no justification for this new field in the physics of the standard model. However, in order to match the density perturbations observed, the universe must have inflated very early on in its evolution—at around Planck scale. Thus, we should be looking to our best quantum-gravity theories for an explanation for how the inflaton might have arisen and for a prediction of its behaviour.

And indeed, as our understanding of string theory and its potential vacua has grown, various models for stringy origin of inflation have been proposed. Or more strictly: most attempts at modelling inflation have depended on string theory's low-energy effective version: supergravity compactified to our observable four dimensions. A number of such models are presented in §3.

However, these models for the inflaton and its potential face a number of problems that remain unresolved. In fact, more often than not, they are ignored by the model-builders. The problems can be categorised as follows [1]:

- *Coupling to gravity*: Unless inflationary potentials are controlled carefully, the inflaton couples to gravity and gains a mass, ending inflation. Some properties required of potentials for the avoidance of this issue are demonstrated in §3.2 and §3.3.
- *Moduli stabilisation*: In most (all?) known supergravity models, a positive cosmological constant in the large dimensions of a compactified theory leads to a running of the moduli and the dilaton and, therefore, the decompactification of the internal manifold. A de Sitter vacuum with stabilised moduli has only been produced using non-perturbative methods beyond supergravity [2]; however, inflation has not yet been successfully modelled in such a setup. These issues are discussed in §4.
- *Cosmological impact of tachyon condensation*: in most stringy inflation models, the exit from inflation occurs owing to tachyon condensation. This has a number of undesirable effects on the cosmology. We shall not discuss this issue here further (see [3] for details).

2 Overview of Inflation for String Theorists

2.1 Problems of Standard Cosmologies

The standard picture of the Big Bang has been very successful at describing our observations of the rate of recession of galaxies, abundance of the light elements in the universe and the cosmic microwave background ("CMB"). However, a number of facts about the observed universe cannot be explained by the standard cosmology. The discussion presented beneath will follow [4].

- *The flatness problem*: The Wilkinson Microwave Anisotropy Probe has confirmed that the universe is essentially flat, with the total density $\Omega_{\text{total}} = 1.02 \pm 0.02$ [5]. When we consider the evolution of the energy density with the scale factor, differentiating the first Friedmann equation:

  $$\frac{\mathrm{d}\Omega}{\mathrm{d}a} = (1+3w)\frac{\Omega_{\text{total}}(\Omega_{\text{total}}-1)}{a}\,, \tag{1}$$

  where w is the proportionality parameter relating the pressure and the energy density in the equation of state of the 'matter' sources, $\rho = wp$.

  For non-relativistic pressureless gas $w = 0$, and $w = 1/3$ for radiation or collections of relativistic particles. Since these two are the only sources of energy density in standard cosmologies, $1 + 3w > 0$ and $\Omega_{\rm total} = 1$ is an unstable fixed point of the differential equation (1). This implies that unless $\Omega = 1$ precisely, it will rapidly deviate from that value. A conservative estimate for the value of $\Omega_{\rm total}$ shortly after the Big Bang necessary to reproduce the observed flatness is

$$0 \leqslant 1 - \Omega_{\rm total} \leqslant 10^{-60} \, ,$$

requiring a tremendous amount of fine-tuning.

- *The horizon problem*: Since only a finite amount of time has passed since the beginning of the universe, photons have only been able to travel a finite distance. Indeed, for a universe that is nearly flat and contains a mixture of matter and radiation, this distance, termed the *horizon distance*, is of the order the inverse of Hubble parameter at the epoch in consideration, with the asterisk denoting the value of the variables at recombination:

  $$d_{\rm hor}(a_*) \sim H_*^{-1}$$

  When we look at the CMB, we are observing the universe at scale factor $a_*/a_0 \approx 1/1200$. The horizon distance at that time subtends approximately 10° in the sky. Therefore, the sky consists of patches that are causally disconnected: no information could have been exchanged between them since the Big Bang; yet they are essentially in thermal equilibrium. How was this achieved?

  It can be shown that this problem is not solved by extending the age of the universe: the angular diameter subtended by the horizon in a decelerating universe actually shrinks with time.

- *Paucity of relics of symmetry breaking*: If we believe that the physics of the standard model was unified in the early universe in a larger gauge group, then the symmetry breaking that must have occurred as the universe cooled would have produced inequivalent vacua in different regions of space and topological defects associated with them. These defects manifest themselves as configurations of energy density and, hence, gravitate: they would have affected structure formation and their effects should be observable; despite searches carried out in the 1990s, no evidence for their existence has been found. A successful cosmological model must be able to explain their dilution beyond the horizon—the standard Big-Bang model does not.

2.2 Chaotic Inflation

A new phase in the evolution of the universe was first proposed by Alan Guth in [6] as a solution of the above problems. Whereas the favourite mechanism by which this process occurs has undergone many modifications, the general idea has remained the same: shortly after its birth, the universe undergoes a period of accelerated expansion, with $\ddot{a} > 0$. This results in the expansion of a small volume of space, which has already been able to achieve causal contact and thermal equilibrium, into all that finds itself within the horizon that we observe today. In the process, any curvature is flattened away and topological defects are diluted. In addition, the quantum fluctuations owing to inflation are expanded to macroscopic size and leave an imprint, compatible with the perturbations observed in the CMB and matter distribution.

The model *du jour* of inflation, *chaotic inflation*, was first proposed by Linde in [7]. The main concepts are summarised beneath.

The essential idea is that a scalar field, ϕ, dominating the energy density of the universe can provide a a negative pressure necessary to drive an accelerating phase, which will slowly evolve and eventually end, if the potential, $V(\phi)$, is chosen appropriately. This field will be called the *inflaton*. By calculating the energy–momentum tensor for the scalar field, it can be shown that it behaves as a perfect fluid with

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{2a}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,, \tag{2b}$$

where we have assumed that the field takes the same value at all points in space by setting its spatial gradient to zero, and the equation of motion

$$H^2 + \frac{k}{a^2} = \frac{1}{3M_{\mathrm{Pl}}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{3}$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\,, \tag{4}$$

where the prime denotes differentiation with respect to ϕ and $M_{\mathrm{Pl}}^{-2} = 8\pi\mathrm{G}$ is the reduced Planck mass, which will subsequently be set to 1. We then impose the *slow-roll* condition, which is satisfied when the kinetic energy of the scalar field, $\dot{\phi}^2/2$, can be neglected in comparison with the potential and $\ddot{\phi}$ is negligible in (4). From (3) and (4) we can then obtain

$$\dot{\phi} \approx -\frac{V'(\phi)}{3H}$$

$$H^2 \approx \frac{1}{3}V(\phi)\,.$$

The above assumptions can be summarised by imposing the following conditions on the *slow-roll parameters*:

$$\epsilon := \frac{1}{2}\left(\frac{V'}{V}\right)^2 \ll 1 \tag{5a}$$

$$\eta := \frac{V''}{V} \ll 1 \tag{5b}$$

where satisfying the first one ensures that the potential is sufficiently flat to guarantee an exponential expansion, whereas the second ensures that the friction term of (4) dominates, allowing the inflationary period to last for a sufficiently long time.

Assuming that the conditions (5) are satisfied, we obtain, from (3), an exponential expansion for the scale factor, i.e. essentially a de Sitter universe, with the value of the potential playing the role of the positive cosmological constant

$$a = a_{\rm i} \exp\left(\sqrt{\frac{V(\phi)}{3}}\, t\right) = a_{\rm i} \exp(Ht)\,.$$

In addition, since $w \approx -1$ for the scalar field in this regime, $1 + 3w < 0$ and $\Omega_{\rm total} = 1$ becomes a stable fixed point: the universe is significantly flattened during this expansion. In the realistic models, when inflation ends, the universe has increased from a size as small as the Planck length $l_{\rm Pl} \sim 10^{-35}$ m to a size of $l \sim 10^{10^{12}}$ m, over a period of 10^{-35} s, which is equivalent to approximately 60 e-foldings [8]. The scalar field starts to oscillate around its minimum decaying to other particles, which equilibrate at some temperature: this is the beginning of the traditional Big-Bang cosmology.

2.3 Hybrid Inflation

In 1991 Linde proposed a modification to chaotic inflation [9], which allows for a graceful end to the inflationary phase and also appears to be much simpler to realise in supersymmetric and string theories. The idea of such models is that there exist two coupled scalar fields. One of them is the inflaton of §2.2. The other has its minimum at zero before and during inflation; however, eventually it becomes tachyonic, gaining a new, non-zero minimum to which it quickly decays. The simplest realisation of this model is [8]:

$$V(\sigma,\phi) = \frac{1}{4\lambda}(M^2 - \lambda\sigma^2)^2 + \frac{m^2}{2}\phi^2 + \frac{g^2}{2}\phi^2\sigma^2\,.$$

For $\phi > \phi_{\rm c} := M/g$ the only minimum of $V(\phi,\sigma)$ lies at $\sigma = 0$. As ϕ slowly decays, eventually it crosses the value of $\phi_{\rm c}$, at which point spontaneous symmetry breaking occurs and a new minimum of V is established: ϕ quickly rolls down to this new value, ending inflation.

3 Inflation from 4D Supergravity

Although many models of the inflaton potential have been proposed, many of which are very successful at describing the observed perturbations in matter distribution, in general they have no strong theoretical motivation: there is nothing in the verified physics of the Standard Model to suggest that such a potential should exist. Therefore, it is natural to attempt to understand what our more basic theories can say about inflation and use such calculations to restrict the space of potential theories.

Over the last 10 years, there has been a significant body of work produced in the attempt to predict inflation-type behaviour from string theory and its effective, low-energy regime—supergravity. This section will present an overview of the type of scalar potentials which are admitted by supergravity in four dimensions and the successful models of inflation that have been found among them.

3.1 Scalars in Supergravity

Theories including supersymmetry are much more restrictive in the possible choices of lagrangians than those without: they are usually written down in terms of the chiral supermultiplet fields Φ, containing all the matter and gauge fields at various component levels. A more detailed exposition of the material summarised here is provided in [10].

In flat space, when gravity is excluded, the most general globally supersymmetric lagrangian that can be written down is of the form:

$$\mathcal{L} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, K(\Phi^i, \bar{\Phi}^j) + \mathrm{Re}\left(\int \mathrm{d}^2\theta\, W(\Phi^i) \right) . \tag{6}$$

$K(\Phi^i, \bar{\Phi}^j)$ is referred to as the Kähler potential, has a power-series expansion in terms of Φ^i and $\bar{\Phi}^j$, is hermitean, and generally encodes the kinetic terms of the fields contained in the supermultiplet. $W(\Phi^i)$ is the superpotential: it is chiral and determines supersymmetry breaking.

When we require that supersymmetry become a local symmetry of the theory, we obtain a theory of supergravity, the bosonic part of which contains Einstein's General Relativity. We can introduce such a model by adding a gravitational term to (6) and exchanging the fermionic coordinates θ for their covariant, curved-space equivalents Θ:

$$\mathcal{L}_{\mathrm{SG}} = -6 \int \mathrm{d}^2\Theta\, \mathcal{E} R + \mathrm{h.c.}\,, \tag{7}$$

where R is the superspace curvature, which contains, among other dynamical fields, the Ricci scalar, while $\mathcal{E}$ is the chiral density of the vielbein, and contains the $\sqrt{-g}$ term as its lowest component—together they give the griviton and gravitino.

The inclusion of (7) changes the required form of the chiral lagrangian for non-minimal models, i.e. ones which will be non-renormalisable. However, these are precisely the models we expect to obtain from string theory consideration, with renormalisability restored by the infinite set of additional states at higher energies. This leads us to the full four-dimensional supergravity langrangian:

$$\mathcal{L} = \int \mathrm{d}^2\Theta\, 2\mathcal{E} \left[\frac{3}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R) \exp\left\{ -\frac{1}{3} K(\Phi^i, \bar{\Phi}^j) \right\} + W(\Phi^i) \right] + \mathrm{h.c.}\,, \tag{8}$$

where $\mathcal{D}$ represents the covariant derivative. Expanded in terms of components at $\Theta = \bar{\Theta} = 0$, (8) yields, *inter alia*, the so-called F-term for the potential of the scalar fields, which are represented by the lowest component of the supermultiplet:

$$V_{\mathrm{F}}(\phi, \bar{\phi}) = \exp(K) \left[(K_{ij^*})^{-1} (D_i W)(D_j W)^* - 3W^* W \right] , \tag{9}$$

where ϕ^i represent the scalar fields, $K_{ij^*} := \partial^2 K/(\partial\phi_i \partial\phi_j^*)$ is the metric on the Kähler manifold and $D_i W := \partial W/\partial\phi_i + W\partial K/\partial\phi_i$. The kinetic part of the lagrangian for the scalars is

$$K_{ij^*}\partial_\mu \phi^i \partial^\mu \bar{\phi}^j \,. \tag{10}$$

A different contribution to the scalar potential can be obtained by adding in a gauge symmetry, resulting in the presence in the theory of vector fields and their superpartners. This modification yields the so-called D-term in the scalar potential:

$$\begin{aligned} V_{\mathrm{D}} &= \frac{g^2}{2}\mathrm{Re}\left(f^{-1}_{AB} D^A D^B\right) \\ D^A &= \frac{\partial K}{\partial \phi_i}(T^A)_i^j \phi_j + \xi^A \,, \end{aligned} \tag{11}$$

where T^A are the generators of the gauge group in the appropriate representation, f_{AB} is the gauge kinetic function, an analytic function of ϕ_i transforming as a symmetric product of two adjoint representations of the gauge group, and the ξ^A are the Fayet-Iliopoulos D-terms, which are non-zero only for $U(1)$ gauge groups.

We then obtain a positive cosmological constant (and therefore inflation) whenever $\langle V \rangle > 0$ at the minimum of the scalar potential. This requires that at least some of the $\langle D_i W \rangle$ or $\langle D^A \rangle$ be non-zero. Thus inflation necessarily breaks supersymmetry. Some of the models proposed for inflation from the above potentials will now be discussed.

It is key to point out that this juncture that the logic of what we are doing is the following. The low-energy effective theory for the string is the ten-dimensional supergravity. There exist only three of these theories, each corresponding to a different string theory. Each of the theories has one scalar—the dilaton—apart from type IIB, which also has the axion. The space is then compactified, according to some scheme, down to four large dimensions. This compactification introduces a number of moduli to the theory, all of which are scalars, each with its own potential. Since the number of different compactification schemes is very large, it is sensible to attempt to restrict them by understanding what is required such that a universe with inflation can result; also we can try to find out whether the compactifications which we know how to carry out and understand well generate appropriate behaviour for the early universe.

3.2 F-Term Inflation

Most initial work on inflation from supersymmetry concentrated on the F-term by assuming that the D-term is flat—and therefore irrelevant—along the inflationary trajectory.

It is argued in [11] that, since (10) requires that, for canonical normalisation, $K =$

$\bar{\phi}_i\phi^i + \ldots$, we can expand (9) (provided $|\phi^i|^2$ is small!) to yield

$$V = \exp\left(\bar{\phi}_k\phi^k + \ldots\right)\left\{(\delta_{ij} + \ldots)\left[\frac{\partial W}{\partial\phi_i} + (\bar{\phi}^i + \ldots)W\right]\left[\frac{\partial\bar{W}}{\partial\bar{\phi}_j} + (\phi^j + \ldots)\bar{W}\right] - 3|W|^2\right\} =$$
$$= \sum_i \left|\frac{\partial W}{\partial\phi^i}\right|^2\left(1 + \sum_j |\phi^j|^2 + \ldots\right) + \ldots = V|_{\phi^i=0} + V|_{\phi^i=0}\sum_j |\phi^j|^2 + \ldots\,,$$

with the second line a result of the fact that $W(\phi^i)$ is a holomorphic function with no constant term in its expansion. The first term in this expansion is the scalar potential for globally supersymmetric theories. This needs to be non-zero for inflation. The second term, a supergravity correction, implies that *all* components of the scalar field gain masses of the order of the first term, spoiling the slow-roll condition (5b). This result is very generic and is a result of the e^K term in in the potential. All F-term inflation theories of this type have to be highly fine-tuned in order for the higher-order terms to cancel the mass corrections in the appropriate way. However, it turns out that string theory provides many potentials for its moduli which result in just such a fine-tuning.

A general form for the Kähler potential which allows for a cancellation of the inflaton mass by higher-order terms is derived in [12]. First, the vector of scalar fields in the theory, ϕ^i, is split into two parts: φ^i, containing the inflaton, and ψ^i. In addition, whenever the components of φ^i excluding the inflaton are required, they will be referred to as χ^i. So, ψ^i and χ^i are constant during inflation, while the inflaton slowly rolls down.

Now, the superpotentials are picked so that

$$W = 0\,, \quad \frac{\partial W}{\partial\varphi^i} = 0\,, \quad \frac{\partial W}{\partial\psi^i} \neq 0\,.$$

This simplifies the scalar potential (9) to

$$V_{\mathrm{F}} = \mathrm{e}^K\,\frac{\partial W}{\partial\psi^i}(K_{ij^*})^{-1}\frac{\partial\bar{W}}{\partial\bar{\psi}^i}\,.$$

If we assume that there are no mixed kinetic terms during inflation, i.e. that $K_{ij^*}|_{\psi^i=0} = 0$. This then implies that we obtain

$$K = A(\varphi,\bar{\varphi}) + B_{ij}(\varphi,\bar{\varphi})\psi^i\bar{\psi}^j + O\big((\psi)^2,(\bar{\psi})^2\big)\,,$$

by performing a Kähler transformation and expanding about $\psi^k = 0$. We can then pick

$$(B_{ij})^{-1} = f(\varphi,\bar{\varphi})(C_{ij})^{-1}(\chi,\bar{\chi})\,, \quad A = -\ln f(\varphi,\bar{\varphi}) + g(\chi,\bar{\chi})\,,$$

where f and g are real functions and C_{ij} is a positive-definite hermitean matrix. Thus, we obtain for our scalar potential

$$V = \mathrm{e}^A\,\frac{\partial W}{\partial\psi^i}(B_{ij})^{-1}\frac{\partial\bar{W}}{\partial\bar{\psi}^j} = \mathrm{e}^{g(\chi,\bar{\chi})}\,\frac{\partial W}{\partial\psi^i}(C_{ij})^{-1}(\chi,\bar{\chi})\frac{\partial\bar{W}}{\partial\bar{\psi}^j}\,,$$

and then the inflaton-masslessness-preserving Kähler potential is required to have the form

$$K = -\ln[f(\varphi,\bar{\varphi}) - \bar{\psi}^i C_{ij}(\chi,\bar{\chi})\psi^j] + g(\chi,\bar{\chi}) + O\big((\psi^i)^2, (\bar{\psi}^i)^2\big) \tag{12}$$

This might not seem like a natural choice, considering that we naďvely might expect a polynomial in ϕ^i, but, as will be demonstrated, this form of the Kähler manifolds arises quite naturally from string-theoretic considerations, as well as being a basis for such theories as no-scale gravity. For example, the Kähler potential of the untwisted sector of the low-energy effective supergravity theory derived from orbifold compactification of superstrings always contains

$$K = -\ln(S+\bar{S}) - \sum_{i=1}^{3}\left(T_i + \bar{T}_i - |\phi_i|^2\right) \tag{13}$$

where S is the dilaton, T_i are the untwisted moduli associated with the radii of compactification and ϕ_i are the untwisted matter fields associated with T_i. If we distribute these scalars as

$$T_1 \in \varphi\,, \quad \phi_1 \in \psi\,, \quad S, T_2, T_3, \phi_2, \phi_3 \in \chi\,, \tag{14}$$

we obtain a form of the Kähler potential compatible with (12).

3.3 D-term Inflation

Since the unwanted inflaton mass arises from the e^K contribution in the F-term, a simpler way to avoid it is to realise the inflation mechanism using the D-term, (11). If we propose that the inflaton is a singlet under the gauge transformations generated by the T^A's, such as is done in [13], then any potential terms involving the inflaton will be automatically annihilated. Thus the gradient of the inflaton potential will only depend on the one-loop radiative correction and will be only logarithmically dependent on the value of the field—and therefore essentially flat. Secondly, the curvature of the potentials for other scalars will depend on their particular gauge charges.

One concrete example of such a realisation is a toy model with a $U(1)$ symmetry including three fields, ϕ_+, ϕ_- and X, with charges $+1$, -1 and 0, respectively. The Kähler potential and f_{AB} are given minimal structure, while the choice of superpotential is the most general trilinear coupling, justifiable by considering R-parity:

$$K = |\phi_-|^2 + |\phi_+|^2 + |X|^2$$
$$f_{AB} = \delta_{AB}\,, \quad W = \lambda X \phi_+ \phi_-\,.$$

The scalar potential can now be evaluated to be

$$V = \frac{g^2}{2}\left(|\phi_+|^2 - |\phi_-|^2 + \xi\right) + \exp\left\{\left(|\phi_-|^2 + |\phi_+|^2 + |X|^2\right)\right\} \times$$
$$\times\, \lambda^2 \left[\, |\phi_+\phi_-|^2\left(1+|X|^4\right) + |\phi_+ X|^2\left(1+|\phi_-|^4\right) + |\phi_- X|^2\left(1+|\phi_+|^4\right) - 3\,|\phi_+\phi_- X|^2 \right].$$

If we fix the value of X, the minimum lies at $\phi_+ = \phi_- = 0$ for $|X| > X_{\rm c} := \frac{g}{\lambda}\sqrt{\xi}$. Indeed, all the terms in the square bracket of the F-term are zero, and hence the exponential term does not contribute and the usual F-term issues are solved. The potential takes the value $V = \frac{g^2}{2}\xi^2$ and the only X-dependence comes from the one-loop correction. For $|X| < X_{\rm c}$, ϕ_+ and ϕ_- become tachyonic causing inflation to end through the hybrid mechanism described in §2.3.

The Fayet-Iliopoulos term, ξ is just a free parameter to be chosen by the model builder. However, if the model is derived from a more precise stringy origin, where the presence of anomalous $U(1)$ symmetries is not unusual, the term can actually be given a fixed value from the Green–Schwarz anomaly-cancellation mechanism (see [14]).

The models presented in this section have summarised the methods used to realise inflation in supergravity- and string-theory-derived scenarios realised directly in the four large dimensions of the compactified manifold. The appearance of inflaton masses of size of the order of the Hubble parameter, spoiling the necessary slow-roll conditions, was a problem for generic potentials, but the existence of models that avoid such an issue without significant fine-tuning was demonstrated. Indeed, we have barely touched the intricacy of model-making that has been achieved: for example in [15], Gaillard *et al.* design a D-term-inflation-based model which is very closely tied to an effective low-energy theory arising from orbifold compactification of string theory with the dilaton stabilised through a gaugino condensate. However, a general problem exists with all approaches starting from supergravity in the already-compactified lower-dimensional space: this will be the main topic considered in §4.

In terms of even more high-tech applications of the above, Kallosh analyses the scalar potentials arising in the D3/D7 model of brane inflation [1]. She concludes that the inflation that arises is an inflation which contains an additional parameter: at one extreme of the parameter, we effectively have F-term inflation, at the other—D-term. Thus even more complex, non-perturbative inflationary scenarios can still reduce to their simple 4D-supergravity models.

4 Scalar Potentials and Supergravity No-Go Theorem

As discussed in §3, the traditional approach for modelling inflation from string theory is to first compactify from the critical dimension down to the observed four, introducing various moduli into the theory, and then search for appropriate potentials for the inflaton. However, this turns out not to be the appropriate way of approaching the problem: a no-go theorem for compactifications of higher-dimensional supergravities to de Sitter spaces stands in the way.

The first issue that we need to consider is the allowed vacuum energy density in those supergravity theories which are actual low-energy limits of string theories, i.e. the ten-dimensional type IIA, IIB and I. These theories are the only possibilities in ten dimensions and each is the effective low-energy description for a different string theory. We have to remember that we cannot add additional chiral matter multiplets *ad hoc* in uncompactified string theories: the only fields allowed are those which are massless states in the string theory. This gives us the dilaton as the only scalar in in IIA and I, while there is also a second scalar in IIB—the axion.

It can be shown that all ten-dimensional supergravity theories, apart from massive type IIA, satisfy the so-called Strong Energy Condition [16]. This is basically a statement that gravity is attractive and can be expressed as

$$\left(T_{MN} - \frac{1}{D-2} g_{MN} T^{L}{}_{L}\right) t^M t^N \geqslant 0 \,,$$

for all non-space-like vectors t^M in D dimensions. This implies that, $\rho \geqslant 0$ and $\rho + 3p \geqslant 0$, where ρ is the energy density and p is the pressure. As demonstrated in (2), this condition is explicitly violated by those stationary scalar fields, where the potential is positive. Thus the consistency with the Strong Energy Condition of $D = 10$ supergravities necessitates that any vacuum value of the scalar field (and hence the cosmological constant) be non-positive, and hence, the space be anti-de Sitter.

Gibbons, [17], appears to have been the first to start from the above assumption and show that it is impossible to get a stable de Sitter space as a result of compactifying a higher-dimensional supergravity theory. However, his exposition is not very pedagogical and we will follow Maldacena and Nuñez [18] in what is essentially the same proof.

The theorem that Maldacena and Nuñez prove states that there are no non-singular wrapped compactification to either de Sitter or Minkowski space with $d \geqslant 2$ non-compact dimensions. We assume:

- Gravity does not contain higher curvature corrections;
- The scalar potential is non-positive, $V \leqslant 0$, i.e. the Strong Energy Condition is satisfied in the non-compactified space;
- Theory contains massless fields with n-form strengths, $F_{I_1 I_2 \ldots I_n}$, with $n < D$. $n = D$ fields are dual to scalars; and
- The d-dimensional effective Newton's constant is finite.

Consider starting with a gravity theory in D-dimensional space (referred to as the "uncompactified" space), with $D - d$ compactified dimensions (the "small" or "internal" dimensions in the "internal" space). Clearly, $D > d$. Let Latin-capital indices denote any of the D dimensions, Latin-minuscule the $D-d$ small directions and Greek-minuscule the d large dimensions in the "remaining" space. Since the resulting manifold is a product space between the internal and large dimensions, there are no cross-terms in the metric and it can be written down as:

$$ds_D^2 = \Omega^2(y) \left(\eta_{\mu\nu} dx^\mu dx^\nu + \hat{g}_{mn} dy^n dy^m\right) \,,$$

where $\eta_{\mu\nu}$ is the metric of the d-dimensional space, either de Sitter or Minkowski, and $\Omega(y)^2\hat{g}_{mn}$ is the metric of the internal space. y represents the compact coordinates, and $\Omega > 0$ is the warp factor. We can write the Einstein equation as:

$$R_{MN} = T_{MN} - \frac{1}{D-2} g_{MN} T^L{}_L$$

We can specify this to the large dimensions and separate the dependence on the non-compact metric, $\eta_{\mu\nu}$, and the internal dimensions:

$$R_{\mu\nu} = R_{\mu\nu}(\eta) - \eta_{\mu\nu}\left(\hat{\nabla}^2 \log \Omega + (D-2)(\hat{\nabla}\log\Omega)^2\right) = T_{\mu\nu} - \frac{1}{D-2}\Omega^2 \eta_{\mu\nu} T^L{}_L\,,$$

where the $\hat{\nabla}$ is the gradient along one of the internal dimensions contracted with the internal metric. Taking a trace with $\eta^{\mu\nu}$ and rearranging the Ω terms yields

$$\frac{1}{(D-2)\Omega^{D-2}}\hat{\nabla}^2\Omega^{D-2} = R(\eta) + \Omega^2 \underbrace{\left(-T^\mu{}_\mu + \frac{d}{D-2}T^L{}_L\right)}_{\tilde{T}} \tag{15}$$

The subsequent proof will demonstrate that the trace of the stress-energy tensor, $\tilde{T}$, must be non-negative and that the Ω term is zero. This implies that the Ricci scalar for the remaining large space cannot be positive:

If we assume that the scalar potential is not varying in time and constant throughout the uncompactified space, then we can write down the stress-energy tensor for the scalars as well as its trace

$$^{\text{Scalar}}T_{MN} = -\frac{V}{2} g_{MN}\,, \qquad ^{\text{Scalar}}\tilde{T} = -V\frac{d}{D-2}\,.$$

Thus, if we assume that $V \leqslant 0$, then $\tilde{T} \geqslant 0$. Similarly the stress-energy tensor for the n-form fields can be found. Maldacena and Nuñez demonstrate that these are positive as well. Therefore, overall, we have a positive trace of the stress-energy tensor.

Since we have assumed that the non-compact cosmology is de Sitter or Minkowski when no matter content is present, then, by Einstein's field equation,

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R + \Lambda\eta_{\mu\nu} = 0 \qquad \Rightarrow \qquad R = \frac{2d}{d-2}\Lambda \geqslant 0\,.$$

This means that, by multiplying (15) by Ω^{2D-4} we get:

$$\Omega^{D-2}\hat{\nabla}^2\Omega^{D-2} \geqslant 0\,, \tag{16}$$

with equality holding only for Minkowski space. If we assume that Ω is bounded below and above in the internal manifold, we can integrate (16) by parts, over the internal space:

$$0 \leqslant \int \mathrm{d}^{D-d}y\,\sqrt{\hat{g}}\,\Omega^{D-2}\hat{\nabla}^2\Omega^{D-2} = -\int \mathrm{d}^{D-d}y\,\sqrt{\hat{g}}\left(\hat{\nabla}\Omega^{D-2}\right)^2 \tag{17}$$

This is only possible if Ω is a constant. Hence the left-hand side of (15) is zero, and since $\tilde{T}$ must be non-negative, $R(\eta) < 0$, so we have a contradiction. The only resolution to this is to allow the space to be anti-de Sitter.

Maldacena and Nuñez also prove, using a similar method, that even is we permit regions where $\Omega \to 0$, we can perform the integration (17) in a restricted region excluding the singularities and prove that the same result holds. Consistent field theories do not exist in spaces which do not have the warp factor bounded from above, so this case is not considered. In addition, they present a separate, slightly different, proof demonstrating that, even though type IIA massive supergravity permits a positive potential in the uncompactified space, a compactification to a either a de Sitter or Minkowski space is still not possible, unless the internal space has boundaries.

The only way to evade these conclusions is to break some of the assumptions presented on page 11. Since we are still attempting to describe inflation in the low-energy regime, we do not wish to bring in the high-energy corrections from string theory if we can avoid it. Thus the only possibility is to introduce scalar fields satisfying the Strong Energy Condition, i.e. add in time dependence for these fields. However, this leads to very serious problems.

In all effective supergravity theories, the scalar field present in the non-compactified theory is the dilaton. Upon compactification, we gain a number of other fields, called moduli, usually representing the sizes of the compactified dimensions. For example, in Calabi–Yau compactification, we obtain a modulus representing the volume of the internal space; in Hořava–Witten M-theory, the modulus represents the distance between the two branes at the fixed points of the orbifolded dimension.

Thus, allowing the scalar fields to vary implies that our moduli are not constant and therefore the sizes of the compactified spaces are varying. This is known as the *moduli stabilisation problem*. In most circumstances it leads to a situation where, in finite time, the modulus runs off to infinity, implying that the internal space decompactifies, and we are left with a ten-dimensional theory. In fact, this problem is so difficult that no solutions for the compactification of ten-dimensional spaces to a stable de Sitter universe involving only supergravity have ever been found [16].

There exist a few methods for improving the situation in the resultant spaces. For example, a non-zero flux of the antisymmetric tensor fields or compactifications with a hyperbolic internal space, i.e. a constant negative curvature, can create a positive potential in the resultant space. However, no solution with a flat enough potential to sustain inflation for an appropriate amount of time has been found. (At most, these are capable of a few e-foldings, compared to the necessary 60.) [16]

When inflation was the only reason to conjure up such a space, this might not have been considered an urgent problem. However, now that the presence of some form of dark energy (i.e. a positive cosmological constant) has been conclusively demonstrated [5], our universe is again evolving toward a de Sitter state, which is likely to last for a very long time. Such a solution needs to be found in the stringy landscape!

Thus it seems that the only appropriate way to proceed is to first identify, in the full string theory, de Sitter vacua which are stable on the cosmological timescale, and only then find appropriate modifications to ensure a sufficient period of inflation. These vacua will undoubtedly involve non-perturbative terms, such as extended sources, i.e. branes.

5 Inflation in Stable de Sitter Vacua?

Since the last section has been, in general, rather negative, we are going to attempt to briefly cover some of the results from modelling inflation in non-perturbative string theory. The general idea is that, since D-branes interact by exchanging closed strings, there is some force which they impart on one another, which will, in effect, depend only on the separation of the branes. This gives us a scalar potential, which we can try to use to model inflation. We leave the details of the string-theoretic calculations required to compute these potentials to another paper in this series [19] and the review [20], and concentrate on the field-theory perspective in the 4D universe.

However, bearing in mind the message of §4, we need to make sure that we choose a compactification of space that has stable moduli. Such a geometry indeed seems to have been constructed [2]: Kachru *et al.* compactify a IIB string theory on a Calabi–Yau manifold in the presence of a three-flux. The moduli are fixed by non-perturbative effects. Finally anti-D3 branes are added, which allows for the breaking of supersymmetry and the elevation of the vacuum to a positive energy. It is claimed that this metastable vacuum should only decay on timescales sufficiently longer than cosmological. The key message for our purposes is that the modulus for this vacuum has Kähler potential and superpotential:

$$K = -3\log(\rho + \bar{\rho}) \tag{18a}$$

$$W = W_0 + A\,\mathrm{e}^{-a\rho}\,, \tag{18b}$$

with ρ complex—its real part of the volume of the Calabi-Yau manifold, while the imaginary is the axion. Computation of the scalar potential according to (9) yields a function with a stable minimum.

Let us now follow [21] in introducing a D3-brane into the six-dimensional Calabi–Yau manifold and investigating whether we successfully obtain appropriate conditions for inflation. The position of the brane will be described by a triplet of complex fields ϕ. Its inclusion modifies our Kähler potential:

$$K = -3\log\left(\rho + \bar{\rho} - k(\phi,\bar{\phi})\right), \tag{19}$$

where the function k is non-trivially related to the Kähler potential for the Calabi–Yau. The definition of ρ also changes slightly, such that the physical volume modulus r is found to be

$$2r = \rho + \bar{\rho} - k(\phi,\bar{\phi})\,.$$

We can now investigate the scalar potential by substituting (18b) and (19) into (9) to obtain

$$V = \frac{1}{6r}\left[\partial_\rho W \partial_{\bar\rho}\bar W\left(1+\frac{1}{2r}\frac{\partial_\phi k \partial_{\bar\phi} k}{\partial^2_{\phi\bar\phi}k}\right) - \frac{3}{2r}\left(\bar W \partial_\rho W + W\partial_{\bar\rho}\bar W\right) + \right] + \frac{D}{(2r)^2}\,. \tag{20}$$

where the additional final term has been included to incorporate the effects of the interaction of the D3-brane with the anti-D3-brane already present in the vacuum; D is a positive constant.

We are now allowed to pick a point in the moduli space where $k(\phi,\bar\phi) = \phi\bar\phi$. In addition, we will assume that ρ and $W(\rho)$ are real. This simplifies (20) to

$$V(\rho,\phi) = \frac{1}{6(\rho-\phi\bar\phi/2)^2}\left[\left(W'(\rho)\right)^2\rho - 3W(\rho)W'(\rho) + \frac{D}{4}\right]\,. \tag{21}$$

We need to make one more change: the kinetic term for the motion of the D3-brane is not pure in the field ϕ, so we need to define a rescaled field to bring the term into canonical form, $\varphi := \phi\sqrt{3/(\rho+\bar\rho)}$. Now if we denote the location of the minimum of (21) as $\rho = \rho_{\rm c}$, (clearly, $\phi = \bar\phi = 0$ there), we get the following potential in the vicinity:

$$V = \frac{V(\rho_{\rm c},0)}{(1-\varphi\bar\varphi/3)^2} \approx V(\rho_{\rm c},0)\left(1+\frac{2}{3}\varphi\bar\varphi\right)\,. \tag{22}$$

Which means that our inflaton, φ, has acquired mass comparable to the value of the potential, violating the slow-roll condition (5b). Thus while the problem of moduli stabilisation remains solved, we have returned to the generic issue of the inflaton's coupling to gravity covered in §3.2.

Despite my presentation here, the class of models which will experience the same issue is actually rather large: it is claimed that the discussion above is generic to any system where the energy density depends on the volume modulus as $r^{-\alpha}$ with $\alpha > 0$, which includes all the form of energy that are known in string theory.

How can we attack this problem? Kachru *et al.* suggest that maybe the superpotential (18b), obtained from the construction of the stable de Sitter vacuum, ought to be modified to include some dependence on ϕ. However, such a setup would require high fine-tuning, such that the mass contributions could cancel exactly.

Another potential avenue for investigation is suggested by [22]: shift symmetry. D-brane BPS states do not experience a net force: contributions from R–R and NS–NS cancel precisely (see [20]). This is the reason by BPS D-branes can be stacked. Thus we can freely change the positions of the branes: the system is symmetric under $\phi^i \to \phi^i + \mathrm{Re}(c^i)$. The way that this symmetry would manifest itself in the model discussed above, would be to change the Kähler potential (19) to one of the form

$$K = -3\log\left(\rho+\bar\rho-\tfrac{1}{2}(\phi-\bar\phi)^2\right)$$

This potential is only dependent on the imaginary part of ϕ, so it is flat in the real direction, offering a potentially massless inflaton. If the symmetry is not exact, but is ever-so-weakly broken, this inflaton could actually gain a mass and drive inflation as needed. However, it is not yet known if the shift symmetry is preserved in the stable de Sitter vacuum proposed.

6 Summary and Conclusion

In this short exposé, we have discussed inflation and how a theoretical grounding from more fundamental physics is being applied to produce the appropriate phenomenology.

Keeping string theory in mind as our best Planck-energy-level theory, we have first introduced supergravity in four dimensions and have attempted to build appropriate potentials in which inflation could be realised. The first stumbling block that we have met is that, in general, such potentials result in giving non-zero masses to all the scalars, including the inflaton, extinguishing the inflationary process prematurely. However, we have demonstrated that such an effect can be eliminated using appropriate Kähler potentials. It appears that such potentials result quite often from string-theoretic compactifications of higher-dimensional supergravities, providing a justification for their non-minimal structures, which absolves us from the charge of unreasonable fine-tuning.

As a next step, we have considered the process of compactification of the low-energy effective field theory arising from the string interactions, i.e. supergravity in ten dimensions, to lower-dimensional manifolds with compact internal spaces. We have found that, since the $D = 10$ supergravities satisfy the Strong Energy Condition, resulting lower-dimensional manifolds can be neither de Sitter nor Minkowski, assuming the scalar fields are stationary. Thus, if we want to only use supergravity, we have to allow the scalar fields to vary with time. Since these scalar fields represent the dilaton and moduli of the compactified dimensions, this effect, called moduli instability, inevitably leads to decompactification and a variation in the coupling strength of the strings. Thus the hope that the models presented in §3 would actually work is rather naïve.

It therefore appears that the only way to proceed is to construct a stable de Sitter vacuum in the non-perturbative regime, where we include extended objects, such as branes. Once obtained, this vacuum can be tested for whether it admits inflation. We do this in §5 and find that it does not, at least in the minimal scenario: the F-term mass-generation process prevents inflation. There are signs that the first working model for string-theory-derived inflation in a stable de Sitter vacuum will soon be found, but so far, none have been produced.

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