Calabi-Yau Compactification

M. Seifert

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1 Introduction

When asked about the ten-dimensional nature of superstring theory, Richard Feynman once replied, "The only prediction [string theory] makes is one that has to be explained away because it doesn't agree with experiment." [8] In a sense, this skepticism is not unfounded, and if string theory did not have so many other desirable features (being a renormalizable theory containing interacting spin-2 particle, for instance), it seems unlikely that it would have gained the cachet it has today. If one wants to be able to apply the elegance of string theory to explain the observed world, however, one has to come to grips with the fact that the world we observe is most emphatically *not* ten-dimensional, or at least remains down to scales of about 10^{-18} metres (the TeV scale, which has been at least indirectly "observed" at particle colliders.)

The usual way to "explain this away", in Feynman's words, is to assume that six of the spatial dimensions of the theory are *compactified*: the spacetime manifold on which the strings move is not an arbitrary ten-dimensional space, but is instead of the form $M^4 \times N^6$ for some compact six-dimensional manifold N^6. This compact manifold is usually taken to have a sufficiently small size as to be unobservable with present technology, and thus we would only see the four-dimensional manifold M^4.

There are a horrendously large number of possible six-dimensional compact manifolds one could choose from, though, and one might hope that the requirement that the four-dimensional theory resemble the observed world would limit our choice of N^6 somewhat. In fact, this turns out to be the case: requiring that supersymmetry is preserved at the compactification scale restricts us to a special class of manifolds known as *Calabi-Yau manifolds.* Further, the topological properties of these manifolds actually determine some of the particle content observed in the low-energy four-dimensional theory; far from being an *ad hoc* assumption necessary to "explain away" undesirable features of string theory, compactification might actually provide an elegant explanation of some features of the observed world.

2 Mathematical Background

To understand the utility of Calabi-Yau manifolds in describing "real physics", one must first understand what a Calabi-Yau manifold is. Unfortunately, this requires a fair deal of rather complicated differential geometry. We will give a brief overview of the essential concepts leading to the definition of a Calabi-Yau manifold, but by necessity this overview will be somewhat sketchy. To fill in the details, the interested reader is referred to [1, Ch. 7], [11, App. D], [12, Ch. 10], and especially Chapters 12 and 15 of [7], which we will follow in much of the exposition below.

2.1 de Rham Cohomology

On an n-dimensional manifold M, we can define a p-form ω without reference to any particular metric on M; see, for example, [1] for details. In terms of a set of local coordinates x^i, we can write ω in terms of a set of components:

$$\omega = \sum_{1 \leq i_1 \leq \cdots \leq i_p \leq n} \omega_{i_1 i_2 \cdots i_p} \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_p} \tag{1}$$

Note that the sum of two p-forms is again a form; thus, the p-forms on an n-dimensional manifold form a vector space. It is fairly obvious from the above expansion that the dimension of this vector space is $\binom{n}{p}$.

We can also define an exterior derivative operator d without respect to a particular metric. This operator takes p-forms to $(p+1)$-forms; in terms of the components of a p-form ω, the components of the $(p+1)$-form $\mathrm{d}\omega$ are given by

$$(\mathrm{d}\omega)_{i_1 i_2 \cdots i_{p+1}} = \frac{1}{p+1} \left(\partial_{i_1} \omega_{i_2 \cdots i_{p+1}} \pm \text{cyclic permutations} \right) \tag{2}$$

where the $\pm$ depends on the sign of the cyclic permutation. If p is odd, this will be an alternating sum; if p is even, we will have a regular sum. This ensures the antisymmetry of the components of $\mathrm{d}\omega$.

It can be shown fairly easily that $\mathrm{d}(\mathrm{d}\omega) = 0$ for any form on M. A p-form for which $\mathrm{d}\omega = 0$ is called a *closed* form, while a p-form for which $\omega = \mathrm{d}\alpha$ for some for some $(p-1)$-form α is called an *exact* form; thus, the statement that $\mathrm{d}(\mathrm{d}\omega) = 0$ is the statement that every exact form is closed. The converse statement, however, is not true; if we have a p-form for which $\mathrm{d}\omega = 0$, we cannot necessarily find a $(p-1)$-form α for which $\omega = \mathrm{d}\alpha$. The archetypical example is when $M = S^1$, parameterized by a coordinate θ, and $\omega = \mathrm{d}\theta$; since the scalar function (i.e. 0-form) θ is not globally well-defined, ω is not exact.

Since everything mentioned thus far can be defined independently of the metric on our manifold M, the properties of the forms which are closed but not exact are purely topological. To quantify these properties, we define two vector spaces: $C_p(M)$, the space of all closed p-forms, and $D_p(M)$, the space of all exact p-forms. Since $D_p(M)$ is a subspace of $C_p(M)$, we can define the quotient space

$$H_p(M) = \frac{C_p(M)}{D_p(M)} \tag{3}$$

More specifically, $H_p(M)$ is the vector space of equivalence classes of closed forms on M, where two closed forms are defined to be equivalent if they differ by an exact form (i.e. $\omega \sim \omega + \mathrm{d}\alpha$.) This new vector space is called the p^{th} *de Rham cohomology group of* M.[1]

The dimension of $H_p(M)$, denoted $b^p(M)$, is called the p^{th} *Betti number of* M. It can be shown that (or, alternately, by definition) the Euler characteristic of our manifold is just the alternating sum of the Betti numbers:

$$\chi(M) = \sum_{p=0}^{n} (-1)^p b^p(M) \tag{4}$$

2.2 Gauge Theories on Manifolds

An n-dimensional differentiable manifold M can be thought of in terms of an "atlas": a set of *charts*, one-to-one functions $\phi_{(\alpha)}$ which map $U_{(\alpha)}$, a region of $\mathbb{R}^n$, to M. These charts must obey two conditions:

[1] More properly, this vector space, isomorphic to $\mathbb{R}^{b^p}$, is called the cohomology group with real coefficients; we can generalize this to cohomology with coefficients in any ring (e.g. $\mathbb{Z}, \mathbb{Z}_n, \mathbb{Q} \ldots$)

- Every point of M must be in the image of at least one chart, and
- The function $\phi^{-1}_{(\beta)} \circ \phi_{(\alpha)}$, where it is well-defined, must be smooth. More precisely: suppose there exists a region of M (call it V) that is mapped to by two charts $\phi_{(\alpha)}$ and $\phi_{(\beta)}$. If we consider the inverse images of this region, $U_{(\alpha\beta)} = \phi^{-1}_{(\alpha)}(V)$ and $U_{(\beta\alpha)} = \phi^{-1}_{(\beta)}(V)$, the map $\phi^{-1}_{(\beta)} \circ \phi_{(\alpha)}$, which takes $U_{(\alpha\beta)}$ to $U_{(\beta\alpha)}$, must be smooth.

Suppose we now want to define a gauge field on M. When we define a gauge theory on $\mathbb{R}^n$, we simply define a set of fields $\psi^i(x)$, and a representation of some group G such that the dimension of the representation is the same as the number of fields. We then postulate that two field configurations are equivalent if they are related by some (position-dependent) element of G, i.e., $\psi'^i(x)$ and $\psi^i(x)$ are equivalent if $\psi'^i(x) = g^i_j(x)\psi^j(x)$ for some $g^i_j(x)$ in the given representation of G. We then define covariant derivatives and and gauge fields so that this assumption holds.

We can try to carry this process over to the case of an arbitrary manifold. The most obvious way would be to use the atlas defining the differentiable structure on M to define a family of fields $\psi^i_{(\alpha)}$, one for each chart $U_{(\alpha)}$ of the atlas. We can further define gauge transformations on each chart as we did before, i.e., two configurations are equivalent on a given chart if they are related by ${\psi'}^i_{(\alpha)}(x) = g^i_{j\,(\alpha)}(x)\psi^j_{(\alpha)}(x)$.

We now run into our first wrinkle: how do we ensure that the fields are "patched together" properly between the charts? We could demand that the fields agree exactly on the overlap between two charts $U_{(\alpha\beta)}$, but this is overly stringent. Rather, since we're postulating the invariance of the physics under gauge transformations, we can be a little more liberal in our matching conditions, and require that the gauge fields in the overlap be related by a gauge transformation of G, i.e.

$$\psi^i_{(\alpha)}(x) = g^i_{j\,(\alpha\beta)}(x)\psi^j_{(\beta)}(x) \qquad \text{on } U_{(\alpha\beta)} \tag{5}$$

for some $g^i_{j\,(\alpha\beta)}(x)$. If we then apply an independent gauge transformation on each chart, the $g^i_{j\,(\alpha\beta)}$'s must also transform:

$$g_{(\alpha\beta)}(x) \to g_{(\alpha)}(x) g_{(\alpha\beta)}(x) g^{-1}_{(\beta)}(x). \tag{6}$$

Similarly, the gauge fields used to define the covariant derivatives must also transform in a particular way between the charts:

$$A_{\mu(\alpha)} = g_{(\alpha\beta)} A_{\mu(\beta)} g^{-1}_{(\alpha\beta)} - (\partial_\mu g_{(\alpha\beta)}) g^{-1}_{(\alpha\beta)} \tag{7}$$

where $g_{(\alpha\beta)}$ is now in the adjoint representation of G. In a very real sense, the transfer functions $g_{(\alpha\beta)}$ determine the structure of the gauge fields on the manifold, just as the overlap functions $\phi^{-1}_{(\beta)} \circ \phi_{(\alpha)}$ determine the differential structure of the manifold itself.

2.3 Chern Classes & Pontryagin Classes

The natural question to ask at this point is whether we can cleverly choose our gauge transformations on each chart to set $g_{(\alpha\beta)}(x) = e$ (the identity) on all of the overlaps. If this is the case, then we can essentially choose a single field ψ^i over all of M instead of having to worry about patching together a family of fields $\psi^i_{(\alpha)}$, and the gauge fields A_μ become similarly independent of the charts.

Unfortunately, this is not always possible, since the topology of the overlap regions (and of the group G) is not always trivial.[2] To quantify this obstruction to the existence of a single gauge transformation for the entire manifold, we turn to the concept of *characteristic classes.* Consider first a $U(1)$ gauge field on M. In the case of an Abelian group G, the gauge field transformations (7) reduce to

$$A_{(\alpha)} = A_{(\beta)} + \mathrm{d}\lambda_{(\alpha\beta)} \tag{8}$$

where we have defined $g_{(\alpha\beta)} = \exp(i\lambda_{(\alpha\beta)})$.

The gauge field $A_{(\alpha)}$ is not, of course, uniquely defined everywhere on M. However, if we apply the d operator to both sides of (8), we find that $\mathrm{d}A_{\mu(\alpha)} = \mathrm{d}A_{\mu(\beta)}$ (since $\mathrm{d}^2 = 0$,) and thus we can uniquely define a field strength $F = \mathrm{d}A$ everywhere on M without worrying about the patching constraints considered above. If we then apply d to the field strength F, we find that $\mathrm{d}F = 0$. In other words, F is a closed two-form, and thus defines an equivalence class in the second cohomology group of M, $H_p(M)$. This equivalence class is known as the *first Chern class* of M (for a given vector bundle structure.)

If we want to extend this notion to a non-Abelian gauge group G, things become a little more complicated. Our gauge field A now takes on values in the Lie algebra of G; we denote this with an extra index on the gauge field, $A^a_{(\alpha)}$, corresponding to the components of A in the algebra.[3] The field strength is now

$$F^a_{(\alpha)} = \mathrm{d}A^a_{(\alpha)} + \frac{1}{2} f^a{}_{bc} A^b_{(\alpha)} \wedge A^c_{(\alpha)} \tag{9}$$

A bit of algebra shows that $F^a_{(\alpha)}$ is no longer gauge-invariant, but instead transforms as

$$F^a_{(\alpha)} = g_{(\alpha\beta)} F^a_{(\beta)} g^{-1}_{(\alpha\beta)} \tag{10}$$

In addition, the "wedge term" in (9) means that F is no longer closed; instead, we have

$$\mathrm{d}F^a = f^a{}_{bc} F^b \wedge A^c \tag{11}$$

While we can't easily define a closed gauge-covariant (or -invariant) two-form in terms of the gauge field, it is possible to define a four-form which is closed:

$$\Omega = \delta_{ab} F^a \wedge F^b \tag{12}$$

This form is uniquely defined over M, and it is easily seen to be closed:

$$\begin{aligned} \mathrm{d}\Omega &= \delta_{ab} \left((\mathrm{d}F^a) \wedge F^b + F^a \wedge (\mathrm{d}F^b) \right) \\ &= f_{acd} \left((F^c \wedge A^d) \wedge F^a + F^a \wedge (F^c \wedge A^d) \right) \\ &= f_{acd} \left(F^c \wedge F^a \wedge A^d + F^a \wedge F^c \wedge A^d \right) = 0. \end{aligned}$$

(The cancellation in the last line occurs because the structure constants are antisymmetric.) Since Ω is closed, it defines an equivalence class in $H^4(M)$, analogous to the first Chern class defined above. If the group G is $SU(N)$, then this class is known as the *second Chern class* of M; if the group G is $SO(N)$, this is known as the *first Pontryagin class* of M.

In general, we can extend this notion further: if there exists an invariant m^{th}-order symmetric tensor $q_{a_1 \cdots a_m}$ in the algebra of G, we can write create an invariant $2m$-form out of it:

$$\tilde{\Omega} = q_{a_1 a_2 \cdots a_m} F^{a_1} \wedge F^{a_2} \wedge \cdots \wedge F^{a_m} \tag{13}$$

This form will again be closed, and thus will define a cohomology class in $H^{2m}(M)$.

[2] A simple example of this is the *Dirac monopole*; see [10, Ch. 10.3].

[3] Remember that a Lie algebra is fundamentally a vector space with some extra structure defined on it.

2.4 Complex Manifolds

In Section 2.2, we defined the concept of a differentiable real manifold. This notion can be extended to the concept of a *complex manifold.* Instead of our charts mapping open sets of $\mathbb{R}^n$ to M, we now define our charts to map regions of $\mathbb{C}^n$ to M. Further, instead of requiring that the "overlap mappings" $\phi^{-1}_{(\beta)} \circ \phi_{(\alpha)}$ must be smooth, we now require that they be holomorphic functions; in other words,

$$\left(\phi^{-1}_{(\beta)} \circ \phi_{(\alpha)}\right)(z, \bar{z}) = \left(\phi^{-1}_{(\beta)} \circ \phi_{(\alpha)}\right)(z). \tag{14}$$

Note that the "complex dimension" of a given manifold will be one-half its "real dimension"; the maps from $\mathbb{C}^n \to M$ can equally well be be thought of as maps from $\mathbb{R}^{2n} \to M$, and holomorphic mappings between open sets of $\mathbb{C}^n$ are necessarily smooth mappings between open sets of $\mathbb{R}^{2n}$. The converse, however, is not true: given an atlas for a real manifold M, it is not always possible to find a complex manifold M' which maps to M under the above correspondence (even if M is even-dimensional.) Thus, a complex manifold can be thought of as a special kind of real manifold, one on which a "complex structure" can be defined. Moreover, this complex structure, should it exist for a given real manifold, need not be unique; one can frequently define multiple complex structures for a given real manifold M which cannot be related to each other by invertible holomorphic functions.

A few examples of complex manifolds are in order here. Obviously, $\mathbb{C}^n$ is a complex manifold. Almost as obviously, we can choose an arbitrary lattice Γ in $\mathbb{C}^n$ and create a quotient space $T^n_{\mathbb{C}} = \mathbb{C}^n/\Gamma$ by identifying points in $\mathbb{C}^n$ differing by a lattice vector. The end result is the complex-manifold analog of the torus; in fact, under the complex-manifold-to-real-manifold mapping described above, its corresponding real manifold is the $2n$-dimensional torus. Changing Γ yields a distinct complex structure; roughly speaking, the "natural" functions between tori corresponding to different choices of Γ will "stretch" the real and imaginary parts of the complex numbers independently, and therefore will not be holomorphic.

Another important example is the complex projective space CP^n. This is defined as a quotient space of $\mathbb{C}^{n+1}$ with the origin removed: given the usual coordinates $Z^1, Z^2, \ldots, Z^{n+1}$ on $\mathbb{C}^{n+1} - \{0\}$, we define two points Z^i and Z'^i to be equivalent if for some complex number $\alpha \neq 0$, $Z'^i = \alpha Z^i$.

In fact, CP^1 is a familiar space in disguise. Considering a point (Z^1, Z^2) on $\mathbb{C}^2$, there are two cases. If $Z^2 \neq 0$, then this point can be identified under rescaling with the point $(Z^1/Z^2, 1)$; if $Z^2 = 0$, we can rescale this point to $(1, 0)$. Thus, we have a space consisting of all the points of the form $(Z^1/Z^2, 1)$, i.e., the complex plane, plus a single point that we approach as $Z^2 \to 0$, i.e., $|Z^1/Z^2| \to \infty$. Thus, CP^1 is the complex plane plus a point at infinity, better known as the Riemann sphere.

A final important family of examples is the set of hypersurfaces in CP^n. To define such a hypersurface, we look at some set of m polynomials P^j in $\mathbb{C}^{n+1}$, and set them all to zero. There are two conditions that these polynomials must satisfy to define a surface in CP^n:

- If a point in $\mathbb{C}^{n+1}$ satisfies $P^j(Z^i) = 0$, we must have $P^j(\alpha Z^i) = 0$ for all non-zero α as well; otherwise, there are points on the surface in $\mathbb{C}^{n+1}$ that get identified with points not on the surface when we go to CP^n, which is nonsensical. The easiest way to assure this is to require that our polynomials be *homogeneous*, i.e., $P^j(\alpha Z^i) = \alpha^{k_j} P^j(\alpha Z^i)$ for some exponent k_j (called the *degree* of P^j.

- If we consider the wedge product $\mathrm{d}P^1 \wedge \mathrm{d}P^2 \wedge \cdots \wedge \mathrm{d}P^m$, this product must not vanish (expect possibly at $Z^i = 0$, which isn't mapped anywhere in CP^n anyways.) This can best be understood in terms of the need for a set of local coordinates. Roughly speaking, if the derivatives and normals of the P^j's are all linearly independent, we can use them as local coordinates for an m-dimensional subspace of $\mathbb{C}^{n+1}$. If not, though, these coordinates can become degenerate and will not necessarily define an $(n+1-m)$-dimensional subspace of $\mathbb{C}^{n+1}$; at such a point, pathologies can creep in.

If these conditions are met, the space so defined can be shown to be a complex manifold; it is called a *complete intersection subspace* of CP^n.

2.5 Kähler Manifolds

Just as we can define holomorphic and antiholomorphic vector fields on $\mathbb{C}^n$, we can define holomorphic and antiholomorphic vector fields on a given complex manifold. The simplest way to see this is to note that in the corresponding $2n$-dimensional real manifold, the tangent vectors at a fixed point transform in the fundamental representation $\boldsymbol{2n}$ of $SO(2n)$. If we look at the way this vector transforms when we go back to the original complex manifold, these should transform in some representation of $U(n)$; under the subgroup decomposition $SO(2n) \to U(n)$, it can be shown that $\boldsymbol{2n} \to \boldsymbol{n} + \bar{\boldsymbol{n}}$, i.e., a real vector field can be thought of as having holomorphic and antiholomorphic which transform independently under $U(n)$. We will denote these components of a given real vector field v^i as v^a and $v^{\bar{a}}$, respectively.

We can further define a metric g_{ij} on a complex manifold. Naïvely, we might think that such a metric could have three different types of components: components which couple holomorphic to holomorphic (g_{ab}), antiholomorphic to antiholomorphic ($g_{\bar{a}\bar{b}}$), and holomorphic to antiholomorphic ($g_{a\bar{b}}$). However, if we want the scalar product of two vectors to transform as a singlet under $U(n)$, the components g_{ab} and $g_{\bar{a}\bar{b}}$ must vanish: these components would yield tensor products $\boldsymbol{n} \times \boldsymbol{n}$ or $\bar{\boldsymbol{n}} \times \bar{\boldsymbol{n}}$, neither of which contain singlets.

A *Kähler manifold* is a complex manifold with additional structure defined. First, the manifold must be endowed with a metric; in addition, we require that everywhere on the manifold, the metric is locally expressible as the derivative of a real scalar function Φ:

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} \Phi(z^a, \bar{z}^{\bar{a}}) \tag{15}$$

This function Φ is called the *Kähler potential*.

Although the condition (15) must hold locally, Φ may or may not be well-defined globally. For example, $\mathbb{C}^n$ is a Kähler manifold with a globally defined Kähler potential $\Phi = z^a \bar{z}^{\bar{a}}$; this function can be carried over onto the complex n-torus $T^n_{\mathbb{C}}$ locally, but since Φ is not single-valued when we identify $\mathbb{C}^n$ under the lattice Γ, Φ is not globally defined. Moreover, our choice of Φ is not unique for a given metric $g_{a\bar{b}}$; we can always shift Φ by an arbitrary holomorphic function plus its conjugate:

$$\Phi'(z^a, \bar{z}^{\bar{a}}) = \Phi(z^a, \bar{z}^{\bar{a}}) + f(z^a) + \bar{f}(\bar{z}^{\bar{a}}) \tag{16}$$

These two new terms will necessarily vanish when we take the derivatives in (15), yielding the same metric $g_{a\bar{b}}$.

It is also worth noting at this point that CP^n can be given a Kähler metric. If we look at the coordinates on $\mathbb{C}^{n+1}$ at a given point, at least one of them is necessarily non-zero; without loss of generality, assume that it is Z^{n+1}. Then we can define coordinates on CP^n in a neighbourhood of this point by $z^a = Z^a/Z^{n+1}$ for $i = 1, \ldots n$; the canonical Kähler metric on this space is then

$$\Phi = \ln\left(1 + z^a \bar{z}^{\bar{a}}\right) \tag{17}$$

It can be shown that this potential yields the unique Kähler metric on CP^n; again, we have a Kähler potential that cannot be globally defined.

Moreover, any complex submanifold N of a Kähler manifold M is necessarily a Kähler manifold: its Kähler potential is simply "inherited" from the Kähler potential on M, with the minor change that we only evaluate the derivatives in (15) in directions tangent to N. This means, in particular, that any complete intersection subspace of CP^n is a Kähler manifold, since CP^n is itself a Kähler manifold.

Finally, we can define a Riemann tensor for a complex manifold, just as we do for a real manifold. This will be a four-index tensor $R_{ijk}{}^l$. If we lower the upper index with the metric, it can be shown [11] that the only non-vanishing components of R_{ijkl} are of the form $R_{a\bar{b}c\bar{d}}$, and that it has the same symmetry properties as the Riemann tensor for a real manifold:

$$R_{a\bar{b}c\bar{d}} = -R_{\bar{b}ac\bar{d}} = -R_{a\bar{b}\bar{d}c} = R_{c\bar{d}a\bar{b}}. \tag{18}$$

Similarly, we can define a Ricci tensor and Ricci scalar by contracting the indices of the Riemann tensor:

$$R_{a\bar{d}} = g^{c\bar{b}} R_{a\bar{b}c\bar{d}} \tag{19}$$

$$R = g^{a\bar{d}} R_{a\bar{d}} \tag{20}$$

2.6 Dobeault Cohomology

The additional structure of a complex manifold M (not necessarily Kähler) allows us to generalize the cohomology groups defined in Section 2.1. We can generalize the p-forms present on a real manifold to a (p, q)-*form*, which is essentially a tensor field $\omega_{a_1 a_2 \cdots a_p \bar{b}_1 \bar{b}_2 \cdots \bar{b}_q}$ on M which is completely antisymmetric in p holomorphic indices and q antiholomorphic indices. We can further define two operators ∂ and $\bar{\partial}$, which take a given (p, q)-form to a $(p + 1, q)$-form or a $(p, q + 1)$-form, respectively; their actions are given by

$$(\partial\omega)_{a_1 a_2 \cdots a_{p+1} \bar{b}_1 \bar{b}_2 \cdots \bar{b}_q} = \frac{1}{p+1} \left(\partial_{a_1} \omega_{a_2 a_3 \cdots a_{p+1} \bar{b}_1 \cdots \bar{b}_q} \pm \text{cyclic permutations of } a_i \right) \tag{21}$$

$$(\bar{\partial}\omega)_{a_1 a_2 \cdots a_p \bar{b}_1 \bar{b}_2 \cdots \bar{b}_{q+1}} = \frac{(-1)^p}{q+1} \left(\partial_{\bar{b}_1} \omega_{a_1 \cdots a_p \bar{b}_2 \bar{b}_3 \cdots \bar{b}_{q+1}} \pm \text{cyclic permutations of } \bar{b}_i \right) \tag{22}$$

The factor of $(-1)^p$ in the definition of $\bar{\partial}$ ensures that the de Rham operator is related to our new operators by $\mathrm{d} = \partial + \bar{\partial}$.

One can define closed and exact (p, q)-forms exactly the same way we did for real forms. Moreover, since $\partial^2 = 0$ and $\bar{\partial}^2 = 0$ via the same antisymmetry arguments we used to show that $\mathrm{d}^2 = 0$, we can define a cohomology group for each one of these operators:

$$H^{p,q}_{\partial}(M) = \frac{\partial\text{-closed } (p, q)\text{-forms on } M}{\partial\text{-exact } (p, q)\text{-forms on } M} \tag{23}$$

$$H^{p,q}_{\bar{\partial}}(M) = \frac{\bar{\partial}\text{-closed } (p, q)\text{-forms on } M}{\bar{\partial}\text{-exact } (p, q)\text{-forms on } M} \tag{24}$$

It can be shown that for a Kähler manifold, these groups are equal. The second of these sets of groups, $H^{p,q}_{\bar{\partial}}(M)$ are called the *Dobeault cohomology groups* of M.

Finally, we can define the analogue of the Betti numbers for these groups. The dimension of $H^{p,q}_{\bar{\partial}}(M)$ (and, in the case of a Kähler manifold, the dimension of $H^{p,q}_{\partial}(M)$) is defined to be the *Hodge number* $h^{p,q}(M)$. Since a given r-form on a real manifold can be written as a sum of

(p,q)-forms with $p+q=r$, it is not entirely surprising that the Hodge numbers are related to the Betti numbers:

$$b^p(M) = \sum_{k=0}^{p} h^{k,p-k}(M) \tag{25}$$

In particular, this means that the Euler characteristic can be written in term of the Hodge numbers:

$$\chi(M) = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q}(M) \tag{26}$$

where n is again the complex dimension of the manifold.

3 Four-Dimensional Physics

3.1 Manifold Structure

In the previous section, we gave an overview of the mathematics necessary to understand Calabi-Yau compactification. The question we have not yet addressed is why any of this is physically useful, a situation which we will try to rectify in this section.

Suppose we have a ten-dimensional supersymmetric field theory, and we want to extract a four-dimensional effective theory from it. The most obvious way to do this would be to postulate that the ten-dimensional manifold our true theory lives on is of the form $M^4 \times N^6$, where M^4 is a four-dimensional spacetime corresponding to the conventional spacetime we know and love, and N^6 is a six-dimensional compact manifold which is too small for us to observe directly.

The obvious question to ask, then, is: what conditions must N^6 satisfy to give us a physically reasonable theory? This question was first addressed by Candelas, Horowitz, Strominger, and Witten [3]. They started with three initial postulates:

1. The manifold M^4 is a maximally symmetric space, i.e. Minkowski, de Sitter, or anti-de Sitter space.
2. Supersymmetry should be unbroken in the resulting $d = 4$ theory.
3. The spectrum of gauge bosons and fermions should bear some passing resemblance to what we see in the real world.

Of these conditions, the second turns out to be the most interesting, but it does seem to fly in the face of common sense; after all, supersymmetry is not a symmetry of the Standard Model. The reason that we want to keep supersymmetry in our compactification scheme is not that we want supersymmetry to persist in the final $d = 4$ theory (we don't), but rather that we don't want to break it at the string scale. One of the original motivations for thinking that supersymmetry is a symmetry of nature (albeit a broken one) is that if it remains unbroken at a sufficiently low energy, it helps to solve the "hierarchy problem" of the Standard Model, namely why the cosmological constant is 120 orders of magnitude less than it should be (see [2] for details.) It is thus desirable to preserve at least some supersymmetry at energies below the string scale.

With the second condition above somewhat more justified, we now examine it in some detail. Consider $N = 1$ supergravity theory in 10 dimensions; this is the low-energy effective field theory limit of either Type I string theory or heterotic string theory. The field content of this theory consists of supergravity multiplet (metric g_{AB}, gravitino ψ_A, two-form B_{AB}, spinor λ, and scalar

ϕ) and a super-Yang-Mills multiplet (field strength F^a_{AB} and spinor χ^a.) A supersymmetry transformation is then generated by a spinor field $\epsilon(x)$ on $M^4 \times N^6$, and the fermionic field variations (up to two-fermion terms) are

$$\delta\psi_A = \frac{1}{\kappa}\nabla_A\epsilon + \frac{\kappa}{32g^2\phi}H_{BCD}\left(\Gamma_A{}^{BCD} - 9\delta^B_A\Gamma^{CD}\right)\epsilon \tag{27}$$

$$\delta\lambda = -\frac{1}{\sqrt{2}\phi}(\nabla_A\phi)\Gamma^A\epsilon + \frac{\kappa}{8\sqrt{2}g^2\phi}H_{ABC}\Gamma^{ABC}\epsilon \tag{28}$$

$$\delta\chi^a = -\frac{1}{4g\sqrt{\phi}}F^a_{AB}\Gamma^{AB}\epsilon \tag{29}$$

where κ is the ten-dimensional gravitational constant, g is the Yang-Mills coupling constant, ∇_A is the covariant derivative, and H_{ABC} is the gauge-invariant three-form associated with B_{AB}. Throughout this section, we will use upper-case Roman indices $A, B, C, \ldots$ to indicate components in the ten-dimensional spacetime; lower-case Roman indices $a, b, c, \ldots$ or $i, j, k, \ldots$ to indicate components in N^6; and Greek indices to indicate components in M^4. Unfortunately, we will also have to use a as a gauge field index; however, we endeavour to avoid the use of a as a gauge field index and a spatial index in the same equation.

We now wish to find a supersymmetric solution of the field equations, i.e., some set of field values for which $\delta\psi_A = \delta\lambda = \delta\chi^a = 0$. To simplify matters, we will also assume that H_{ABC} vanishes, and that the dilaton ϕ is constant;[4] under this assumption, the conditions that the solution be supersymmetric are simply

$$\nabla_A\epsilon = 0 \quad \text{and} \quad F^a_{AB}\Gamma^{AB}\epsilon = 0 \tag{30}$$

for some non-zero spinor ϵ.

The first of these two equations has two very important consequences. First, it implies that[5]

$$[\nabla_A, \nabla_B]\,\epsilon = R_{ABCD}\Gamma^{CD}\epsilon = 0. \tag{31}$$

Since we are postulating that M^4 is maximally symmetric, the Riemann tensor for this space is of the form $R_{\mu\nu\rho\tau} = K(g_{\mu\rho}g_{\nu\tau} - g_{\nu\rho}g_{\mu\tau})$; applying this to (31) implies that $K\Gamma_{\mu\nu}\epsilon = 0$ for all values of μ, ν. Thus, $K = 0$, and we conclude that M^4 must be Minkowski space.

The second consequence is perhaps the more interesting one. Restricting attention to components in N^6, we have $\nabla_i\epsilon = 0$ everywhere on the manifold, i.e., there must exist a globally defined covariantly constant spinor field on N^6. Certainly one can define such a field in any given neighbourhood; however, it is not at all evident that one can do so globally. As evidence that such a spinor field may not be possible, we note that the analogous *vector* field case on S^2 is known to be impossible; the celebrated "hairy ball" theorem states that any vector field on a sphere (i.e., "hairs" on a "ball") must vanish at at least one point, and if a covariantly constant vector field vanishes at one point it must vanish everywhere. In contrast, a global covariantly constant vector field obviously exists on the torus. It is evident from these examples, then, that the existence of such globally defined fields is a topological restriction on N^6.

To better quantify this restriction, we examine the actions of *holonomies* on N^6. Suppose we take a spinor ϵ defined at a given point p on N^6 and, using the spin connection on N^6, parallel-transport it around some closed curve γ. This will define a new spinor ϵ' which is related to ϵ

[4]The full analysis, with non-vanishing H_{ABC} and varying ϕ, is given in [3]; the results mentioned here all carry over to this case.

[5]A similar integrability condition was derived in class in the presence of a cosmological constant.

by some $SO(6)$ transformation depending solely on the loop γ:

$$U_\gamma = \mathrm{Pexp} \oint_\gamma \omega_i \, \mathrm{d}x^i, \tag{32}$$

where Pexp is the path-ordered exponential and ω_i is the spin connection. Since the groups $SO(6)$ and $SU(4)$ are isomorphic, we can equally well view this as an $SU(4)$ transformation; this has the advantage that the spinor representations $\mathbf{4}$ and $\mathbf{4'}$ of $SO(6)$ are equivalent to the fundamental and antifundamental representations $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $SU(4)$. Thus, demanding that our spinor be left invariant under the action of *any* holonomy is equivalent to demanding that a given complex four-vector, when acted on by $SU(4)$, is left invariant. We conclude that for a non-trivial covariantly constant spinor field to exist globally on N^6, all holonomies on the manifold must belong to an $SU(3)$ subgroup of $SO(6)$ (or $SU(4)$.) Such a manifold is commonly called a *manifold of* $SU(3)$ *holonomy.* It is not too hard to show that a metric has $SU(3)$ (or, more generally, $SU(n)$) holonomy if and only if it is Ricci-flat (see, e.g., [7, Ch. 15].)

What class of manifolds, then, admit a metric of $SU(3)$ (or, more generally, $SU(n)$) holonomy? We already know part of the answer: the complex structure of a Kähler manifold guarantees that holomorphic vectors will be mapped to holomorphic vectors under any given holonomy (this is not entirely obvious, but is not too hard to show from the structure of the Riemann tensor.) Thus, a Kähler manifold necessarily has a $U(n) = SU(n) \times U(1)$ holonomy.

The question then becomes what conditions must be imposed such that this $U(1)$ factor is trivial. The answer recalls our earlier discussion of Chern classes: the spin connection is nothing more than a gauge field on N^6, and the $U(1)$ factor nothing more than its Abelian part. If this Abelian gauge field can be entirely "gauged away", i.e., if the corresponding F is zero, then the $U(1)$ factor is trivial and the $U(n)$ factor becomes an $SU(n)$ factor. But since F defines the first Chern class of the manifold, this class must be trivial.

This means that if we have a Kähler manifold of $SU(n)$ holonomy, its first Chern class must vanish. The converse of this theorem, that any Kähler manifold whose first Chern class vanishes admits a metric of $SU(n)$ holonomy, was first conjectured by Calabi and proved (twenty years later) by Yau. Such a manifold is frequently called a *Calabi-Yau manifold.* This theorem greatly simplifies the search for metrics of $SU(n)$ holonomy; manifolds of vanishing first Chern class can, in general, be found more easily than Ricci-flat Kähler manifolds.

3.2 Massless Particle Content

One of the more appealing features of Calabi-Yau compactification is that it allows us to predict the number of chiral generations present in the four-dimensional theory from very general considerations; we will give a brief outline of these now. Our outline here will roughly follow that of [9].

Consider the effective field theory that emerges from the heterotic string: a supergravity multiplet and an $E_8 \times E_8$ Yang-Mills supermultiplet.[6] The group E_8 has a maximal subgroup of $SU(3) \times E_6$, under which the adjoint decomposes as

$$\mathbf{248} \to (\mathbf{1}, \mathbf{78}) + (\mathbf{3}, \mathbf{27}) + (\bar{\mathbf{3}}, \overline{\mathbf{27}}) + (\mathbf{8}, \mathbf{1}) \tag{33}$$

The $SU(3)$ gauge field can be identified with the spin connection on our Calabi-Yau manifold N^6; in other words, we consider our spin connection to be *embedded* in our gauge group. This

[6] The $SO(32)$ Yang-Mills supermultiplet that emerges from the type I string theory is less phenomenologically desirable; see [3] for details.

fact has important implications—in particular, it allows the theory to be anomaly-free—but in the interests of space we will not discuss these here. The interested reader is referred to [3, 7].

Under this decomposition, the gauge fields will decompose into components tangent to M^4 and N^6, denoted A^a_μ and A^a_i, respectively; the index a here denotes the gauge group index. The fields A^a_i will further decompose into holomorphic and antiholomorphic parts A^a_i and $A^a_{\bar{i}}$. It is important to note here that the transformations of the $SU(3)$ part of the gauge field (i.e., the index a) are the *same* as those of the N^6 indices $i, \bar{i}$, since the $SU(3)$ factor of the gauge group is simply the spin connection. Thus, a gauge field component tangent to N^6 and transforming non-trivially under the $SU(3)$ of the gauge group can be viewed as a tensor in N^6.

With this in mind, we examine each of the components A^a_i and $A^a_{\bar{i}}$:

- Components transforming as $(\mathbf{1}, \mathbf{78})$. In this case, the "naïve" vector index is the only N^6 index present; the components A^a_i and $A^a_{\bar{i}}$ can be viewed as $(1,0)$- and $(0,1)$-forms, respectively. Note that these forms are conjugate.
- Components transforming as $(\mathbf{3}, \mathbf{27})$. Consider the antiholomorphic component $A^a_{\bar{i}}$ first. This can be viewed as a $(1,1)$ form $\tilde{A}_{j\bar{i}}$ on N^6, since it has a holomorphic index from the $\mathbf{3}$ of the gauge group and an antiholomorphic vector index.

  The holomorphic component A^a_i is a little more subtle. This has two holomorphic indices, one from the gauge group and one from the vector index, and could be written as $\tilde{A}_{ij}$. However, the indices i and j are not antisymmetrized, so $\tilde{A}_{ij}$ is not a form. We can fix this, though, by using the metric and the anti-symmetric $SU(3)$ tensor $\Psi_{\bar{i}\bar{j}\bar{k}}$ to construct a $(1,2)$ form out of $\tilde{A}_{ij}$:
$$\tilde{A}_{i\bar{j}\bar{k}} = \tilde{A}_{ij} g^{j\bar{l}} \Psi_{\bar{l}\bar{j}\bar{k}} \tag{34}$$
- Components transforming as $(\bar{\mathbf{3}}, \overline{\mathbf{27}})$. The analysis for these components is practically identical to that for the components transforming as $(\mathbf{3}, \mathbf{27})$; we will end up with a $(1,1)$-form from the holomorphic component A^a_i and a $(2,1)$-form from the antiholomorphic component $A^a_{\bar{i}}$. Moreover, these forms will be conjugate to the forms we found in the previous case.
- Components transforming as $(\mathbf{8}, \mathbf{1})$. These will be uncharged under the E_6, and thus can acquire masses quite easily when we consider the effective low-energy theory; we will therefore ignore them.

What does all this have to do with the massless spectrum? Recall that under a general compactification of a scalar field Φ, the wave operators decouple into pieces acting in the "physical" space and in the compactified space:

$$\Box = \nabla_\mu \nabla^\mu + \nabla_i \nabla^i \tag{35}$$

If the compact manifold has a discrete set of eigenvalues of $\nabla_i \nabla^i$ (as it will in general), then we can decompose the scalar function into eigenmodes. Of course, the coefficients of these modes will be dependent on the position in the physical spacetime, i.e.,

$$\Phi(x^\mu, x^i) = \sum_\Lambda \phi_\Lambda(x^\mu) \psi_\Lambda(x^i) \tag{36}$$

where Λ is an index running over the eigenfunctions $\psi_\Lambda(x^i)$. The four-dimensional scalar fields ϕ_Λ will then decouple and become massive, obeying the equation

$$(\nabla_\mu \nabla^\mu + m^2_\Lambda) \phi_\Lambda = 0 \tag{37}$$

where m_Λ^2 is the eigenvalue of ψ_Λ under the operator $\nabla_i \nabla^i$. We see, then, that for a given field to remain massless under a compactification, it is necessary that there exist a non-trivial zero-mode of the field on the compactified space. This result can be carried over to arbitrary form fields and spinor fields as well, although it takes some care to properly define a Laplacian for such fields; see [12] for details.

Returning to our Calabi-Yau compactified theory: we found that we have components of the gauge field which behave as $(1,0)$-forms, $(1,1)$-forms, and $(2,1)$-forms (plus their conjugates.) The natural question to ask, then, is how many independent zero-modes of such fields exist on a given manifold M. We will simply cite the result here that the number of linearly independent zero-modes of a (p,q)-form is precisely the Hodge number $h^{p,q}(M)$. Thus, the number of massless fields (and their superpartners) is solely a function of the topology of the compactification manifold N^6.

We can carry this analysis one step further. We have a set of $h^{2,1}$ massless chiral fields (corresponding to the $(2,1)$-forms) which transform in the $\mathbf{27}$ of E_6, and a set of $h^{1,1}$ massless chiral fields which transform in the $\overline{\mathbf{27}}$ of E_6. When we go to the low-energy theory, the chiral fields can easily couple to the anti-chiral fields through renormalization group effects and thereby acquire effective masses; thus, the number of chiral fields remaining in the four-dimensional theory will be

$$\mathcal{N} = |h^{2,1} - h^{1,1}| \tag{38}$$

Further, it can be shown that $h^{1,0} = 0$ for a manifold of $SU(3)$ holonomy; thus, we will not have any massless modes transforming in the $\mathbf{78}$ (adjoint) of E_6.

This leads us to a rather surprising result. It can be shown that the Hodge numbers obey certain identities; for a three-complex-dimensional manifold,

$$h_{p,q} = h^{3-p,3-q} = h^{q,p}, \tag{39}$$

and

$$h_{p,0} = h^{3-p,0}. \tag{40}$$

It is fairly obvious that $h^{0,0} = 1$ (a $(0,0)$-form is closed if its derivative is zero, which can only occur if it is constant.) Using the above symmetries, this implies that there are very few independent Hodge numbers:

$$\begin{aligned} h^{0,0} = h^{3,0} = h^{0,3} = h^{3,3} = 1 \\ h^{1,1} = h^{2,2} \\ h^{2,1} = h^{1,2} \end{aligned} \tag{41}$$

and all others zero. Thus, if we write out all sixteen terms in (26), we find the surprising result that

$$|\chi(M)| = 2|h^{1,1} - h^{2,1}| = 2\mathcal{N}. \tag{42}$$

In other words, the number of chiral multiplets present in the low-energy theory is precisely one-half the absolute value of the Euler characteristic of the compactified space. In most E^6 unified theories, one embeds a single generation of quarks and leptons into a single $\mathbf{27}$ of E^6; thus, this result can be viewed as saying that the number of generations of quarks and leptons we observe is a direct consequence of the otherwise unobservable Calabi-Yau manifold used to compactify the ten-dimensional theory. This is a very elegant result, and accounts for much of the excitement surrounding the heterotic string and Calabi-Yau compactification in the late '80s.

4 Phenomenologically Realistic Compactifications

Now that we have established the properties of Calabi-Yau compactifications, we come to the hard part: trying to construct models that agree with the real world. We saw in Section 3.1 that compactifying a ten-dimensional theory on a Kähler manifold of $SU(3)$ holonomy—or, equivalently, a Kähler manifold of vanishing first Chern class—allows us to preserve supersymmetry below the string scale. We further saw in Section 3.2 that the massless spectrum of a given theory is intimately related to the topological properties of the compactified directions.

What we did not actually show in these previous two sections was how to describe a Calabi-Yau manifold in order to extract these topological properties. This was not entirely an oversight on our part. While the number of Calabi-Yau manifolds with one or two complex dimensions is known (the torus $T^1_{\mathbb{C}}$ is the unique one-dimensional compact Calabi-Yau manifold; in two complex dimensions, there is the product of two tori $T^2_{\mathbb{C}}$ and another space called $K3$), the situation in three complex dimensions is much different. Several thousand Calabi-Yau three-folds have been discovered; with one exception ($T^6_{\mathbb{C}}$), their metrics are not explicitly known, and thus very little non-topological information (which is necessary to determine such things as coupling strengths) can be extracted from them; and it is not even known (although it is strongly suspected) that the number of topologically distinct Calabi-Yau three-folds is finite. With this in mind, we will describe a particular method of constructing Calabi-Yau spaces; the reader is cautioned here that the manifolds presented here are only a tiny fraction of what is known about specific Calabi-Yau manifolds, and an even tinier fraction of what actually exists.

4.1 Complete Intersection Calabi-Yau Spaces

In Section 2.5, we introduced the complex projective plane CP^n and noted that it was a Kähler manifold. We also described the complete intersection subspaces of CP^n, which were defined via the vanishing of a set of homogeneous polynomials P^i within CP^n. The obvious question one can ask, then, is whether these complete intersection manifolds can be Calabi-Yau manifolds.

The answer, as it turns out, is yes, subject to certain constraints on the polynomials. Suppose we attempt to define an n-dimensional manifold via the complete intersection of m polynomials $P^1, P^2, \ldots, P^m$ in the complex projective space CP^{n+m}. Suppose further that these polynomials have degrees $k^1, k^2, \ldots k^m$. Then it can be shown that such a space has $SU(N)$ holonomy if and only if

$$\sum_{i=1}^{m} k^i = n + m + 1. \tag{43}$$

A sketch of the proof is as follows: on a given complex manifold of dimension n, we can construct a completely antisymmetric, completely holomorphic tensor $\epsilon_{a_1 a_2 \cdots a_n}$ by taking the completely antisymmetric part of the tensor product

$$\underbrace{\boldsymbol{n} \times \boldsymbol{n} \times \cdots \times \boldsymbol{n}}_{n \text{ times}}$$

If we attempt to parallel-propagate this tensor product using a holonomy, as described in Section 3.1, it will be left invariant only if the holonomy group is $SU(n)$; a more general $U(n)$ holonomy group will not leave ϵ invariant. Thus, if we can find a covariantly constant $(n,0)$-form on our space, our space is a Calabi-Yau manifold. The question is then reduced to finding a covariantly constant $(n,0)$-form on our complete intersection subspace. This is easy to do in a region of the subspace where a given coordinate (say, z^1) is non-zero. Attempting to continuously extend this form to the point where z^1 is zero, we find that the coordinate transformations give us a factor of $(z^1)^{n+m+1-\sum k_i}$. If this form is to neither vanish nor diverge we are led to the condition (43). A more complete account of this proof can be found in [7].

The condition (43) is quite restrictive on the possible sets of polynomials one can use to find a complete intersection Calabi-Yau manifold (often abbreviated to a *CICY manifold.*) We note that a homogeneous polynomial of degree one is simply a linear polynomial, and it is not too hard to show that this simply pick a space isomorphic to CP^{n-1} out of CP^n. Thus, we need only restrict our attention to polynomials with degree two or greater. To find a CICY manifold, then, we need to find a set of m integers greater than one that add up to $m+n+1$. The possible lists of integers for $n=3$, the case of interest, are

$$\{5\}, \{4,2\}, \{3,3\}, \{3,3,2\}, \text{ and } \{2,2,2,2\}.$$

These will be subspaces of CP^4, CP^5, CP^5, CP^6, and CP^7, respectively. We will denote these spaces using the conventional notation $Y_{n+m+1;k_1,k_2,\ldots,k_m}$. The Euler characteristics of such spaces can be calculated through cutting-and-pasting arguments (see [7];) the result is

$$\begin{aligned} \chi(Y_{4;5}) &= -200 \\ \chi(Y_{5;4,2}) &= -176 \\ \chi(Y_{5;3,3}) &= -144 \\ \chi(Y_{6;3,2,2}) &= -144 \\ \chi(Y_{7;2,2,2,2}) &= -128 \end{aligned} \tag{44}$$

One might wonder at this point what the point of describing such manifolds is; there seems to be a very limited class of them (only five), and they give rise to horrendously unrealistic physics (in light of (42), between 64 and 100 generations of quarks and leptons!) The second of these questions will have to wait for the next section, but we can answer the first with a simple generalization of the CICY manifolds.

Suppose, instead of looking at a subspace of CP^n, we use as our "ambient space" an arbitrary product of complex projective spaces $CP^{n_1} \times CP^{n_2} \times \cdots \times CP^{n_F}$. We can then write down a set of polynomials and examine the subspace created by setting them to zero. These polynomials need only be homogeneous within each CP^n factor; in other words, if X^i and Y^i are coordinates for two of the CP^{n_i} factors, we need only have $P(\alpha X^i, \beta Y^i) = \alpha^k \beta^l P(X^i, Y^i)$. This case was thoroughly examined by Candelas, Dale, Lütken, and Schimmrigk [4]; using a computer, they compiled a list of 7868 possible sets of ambient spaces and polynomial degrees, each one of which could (in theory) define a distinct Calabi-Yau manifold. Further research by Green, Hübsch, and Lütken [6] showed that at least 2590 of these are topologically distinct. Thus, this enlarged class of CICY manifolds is much more interesting than the first set of five we derived.

4.2 Quotient Spaces and Realistic Models

While the work in [4] reassures us that the class of CICY manifolds is not quite as limited as one would naïvely think, it does nothing to solve the proliferation of fermionic fields seen in these models. Indeed, the Euler characteristics of the list of manifolds in [4] were calculated in the same paper; these numbers included every even integer from 0 to -120, with four exceptions: -2, -6, -10, and -118. (Also on the list were 13 even integers between -122 and -200.) The absence of -6 is particularly troubling, since this would correspond to three generations of fermions in the four-dimensional compactified theory. How, then, can we resolve this dilemma?

To answer this, let us return to the simplest CICY examined thus far, $Y_{4;5}$. This manifold is defined by a fifth-order polynomial in the $\mathbb{C}^5$ coordinates on CP^4, e.g.,[7]

$$P = \sum_{i=1}^{5} Z_i^5 = 0 \tag{45}$$

This polynomial (and hence the surface defined by it) has a good deal of symmetry; in particular, it is invariant under the discrete $\mathbb{Z}_5 \times \mathbb{Z}_5$ symmetry generated by

$$\begin{aligned} a &: (Z_1, Z_2, Z_3, Z_4, Z_5) \to (Z_2, Z_3, Z_4, Z_5, Z_1) \\ b &: (Z_1, Z_2, Z_3, Z_4, Z_5) \to (Z_1, e^{2\pi i/5} Z_2, e^{4\pi i/5} Z_3, e^{6\pi i/5} Z_4, e^{8\pi i/5} Z_5) \end{aligned} \tag{46}$$

An important fact about this group transformation is that it acts *freely*, i.e., it has no fixed points. Thus, the quotient manifold $Y_{4;5}/(\mathbb{Z}_5 \times \mathbb{Z}_5)$ is well-defined.[8] Moreover, it can be shown that this group action preserves the $SU(3)$ holonomy of $Y_{4;5}$; in general, this will be true of any discrete group acting freely on a manifold of $SU(3)$ holonomy.

The reason to consider such group actions is this: the Euler characteristic can be viewed as the integral of a curvature invariant (dependent on the dimension of the manifold). If we take the quotient space of a manifold by a freely acting group and compute this same integral, we will effectively be integrating the same curvature invariant over a fraction of the space, and this fraction will be equal to the order of the discrete group. In other words, if a discrete group F freely acts on a manifold M, we will have

$$\chi(M/F) = \frac{\chi(M)}{|F|} \tag{47}$$

where $|F|$ is the order of the discrete group. In the case of $Y_{4;5}/(\mathbb{Z}_5 \times \mathbb{Z}_5)$, this implies that the Euler characteristic of our new manifold is $-200/25 = -8$, and compactifying on it instead of $Y_{4;5}$ would give us four generations of fermions instead of 100.

One might respond that this is all very well and good, but four generations are not observed in the real world. Can we take one of the manifolds found in the list in [4] and quotient it by some discrete group to get a manifold of Euler characteristic -6? This question was examined in a follow-up paper [5]. Of the manifolds on their list, they were able to eliminate all but three of them from consideration; of these three, one had Euler characteristic -18 and the other two had Euler characteristic -48. The first of these manifolds is known as the *Tian-Yau* manifold; it consists of a subspace in $CP^3 \times CP^3$ given by three polynomials:

$$\begin{aligned} P_1 &= \sum_{i=1}^{4} X_i^3 = 0 \\ P_2 &= \sum_{i=1}^{4} Y_i^3 = 0 \\ P_3 &= \sum_{i=1}^{4} X_i Y_i = 0 \end{aligned} \tag{48}$$

[7] This choice of polynomial may seem somewhat arbitrary; why not choose, say, a $Z_i^2 Z_j^3$ term as well? It is not too hard to show, though, that two different sets of polynomials of the same degrees will give topologically equivalent hypersurfaces; the only difference will be in the resulting complex structures.

[8] Fixed points in the group action would lead to orbifold singularities. These singularities can be patched up via a process known as "blowing up" the singularities, or they can be left in, creating a Calabi-Yau orbifold. Both options are interesting, but space does not permit their discussion here.

where X_i and Y_i are the $\mathbb{C}^4$ coordinates of the two CP^3's. This surface admits a free action of $\mathbb{Z}_3$,

$$(X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4) \to (e^{2\pi i/3} X_1, X_2, X_3, X_4, e^{4\pi i/3} Y_1, Y_2, Y_3, Y_4) \qquad (49)$$

Thus, under the above construction, the quotient space $M/\mathbb{Z}_3$ has Euler characteristic -6, and it is therefore possible to construct a model containing three generations of quarks and leptons out of string theory.[9]

5 Conclusion

If we demand that supersymmetry be preserved when we compactify our ten-dimensional theory to four dimensions, we are led (via a fair amount of complicated mathematics) to the requirement that our compact six-dimensional manifold be a Calabi-Yau manifold. The particle content of the resulting four-dimensional theory is intimately related to the topological properties of our compactification manifold; in particular, we have the very elegant result that the number of generations present in the four-dimensional theory is one-half the Euler characteristic of the compactification space.

We should not get too proud of ourselves at this point. While the prediction that the Euler characteristic of the compactification space is related to the number of generations is a very elegant and striking one, we must keep in mind that other compactification schemes could have predicted almost any number of generations between 0 and 60—and this is only considering the simplest class of Calabi-Yau manifolds. This non-uniqueness is somewhat inelegant in and of itself. It is certainly true that string theory in ten dimensions has no "tuneable parameters", but there remains the choice among tens of thousands of compactification manifolds, all with potentially distinct four-dimensional physics; and as of yet, the theory does seem to exhibit a preference for any one of them. Moreover, the full properties of Calabi-Yau manifolds are not known, and it is not yet possible to extract a full four-dimensional low-energy theory from any compactification scheme other than the complex 3-torus $T^3_{\mathbb{C}}$. Nonetheless, the properties described here are quite compelling; while we should not view Calabi-Yau compactification as the be-all and end-all of physics, it certainly qualifies as evidence that string theory is on the right track.

References

[1] Arnold, V. I. *Mathematical Methods of Classical Mechanics*, second edition. Springer-Verlag, New York, 1989.

[2] Bailin, David, and Alexander Love. *Supersymmetric Gauge Field Theory and String Theory.* Institute of Physics Publishing, Philadelphia, 1994.

[3] Candelas, P., et al. "Vacuum configurations for superstrings." Nuc. Phys. **B258**, 46–74 (1985).

[4] Candelas, P., et al. "Complete intersection Calabi-Yau manifolds." Nuc. Phys. **B298**, 493–525 (1988).

[9] As for the two manifolds of Euler characteristic -48, the authors of [5] were unable to explicitly construct a free group action of order eight on them. They were, however, able to show that this group must be $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ if it exists.

[5] Candelas, P., C. A. Lütken, and R. Schimmrigk. "Complete intersection Calabi-Yau manifolds (II): Three generation manifolds." Nuc. Phys. **B306**, 113–136 (1988).

[6] Green, P., T. Hübsch, and C. A. Lütken. "All the Hodge numbers for all Calabi-Yau complete intersections." Class. Quant. Grav. **6**, 105–124 (1989).

[7] Green, Michael B., John H. Schwarz, and Edward Witten. *Superstring Theory*, Vol. 2. Cambridge University Press, Cambridge, UK, 1987.

[8] Mlodinow, Leonard. *Feynman's Rainbow.* Warner Books, New York, 2003.

[9] Polchinski, Joseph. *String Theory*, Vol. 2. Cambridge University Press, Cambridge, UK, 1998.

[10] Ryder, Lewis H. *Quantum Field Theory*, second edition. Cambridge University Press, Cambridge, UK, 1996.

[11] Wess, Julius and Jonathan Bagger. *Supersymmetry and Supergravity*, second edition. Princeton University Press, Princeton, NJ, 1992.

[12] von Westenholz, C. *Differential Forms in Mathematical Physics*, revised edition. North-Holland, New York, 1981.