# Duality and the Unification of String Theory

Yi Zheng

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## 1 Introduction

As the title says, I will first introduce the notion of duality and general procedures of testing dualities. Then I will discuss several important examples of dualities and use those examples to attempt a definition of M-theory, which can serve as the candidate for the unified theory. I also give a brief introduction to discrete light cone quantization (DLCQ), which could serve as a starting point of a possible future in-depth discussion of M-theory. In the last section, I highlight several references from which I benefit a lot.

## 2 Notion of duality

There are five consistent 10d string theories: IIA, IIB, I , Heterotic SO(32), Heterotic $E_8 \times E_8$. We can also get many different string theories by compactifying those five on appropriate manifold $\mathcal{M}$(*e.g.* tori of different dimensions, K3, Calabi-Yau manifolds etc.). Each of them is parametrized by moduli:

- String coupling constant (vev of the dilaton field $\langle\phi\rangle$),
- Shape and size of $\mathcal{M}$ (information on the metric),
- Various other background fields (*e.g.* Wilson lines).

Inside the moduli space of the theory there is a certain region where the string coupling is weak and perturbation is valid. Elsewhere the theory is strongly coupled.

String duality provides us with an equivalence map between two different string theories. In general this equivalence relation maps the weak coupling region of one theory to the strong coupling region of the second theory and vice versa (this is called S duality). Under duality, typical perturbation expansions get mixed up. Thus for example, tree level results in one theory might include perturbative and non-perturbative corrections in the dual theory. Also under duality, many of the elementary string states in one theory get mapped to solitons and their bound states in the dual theory.

There are special cases where the situation is a bit different:

- Self-duality: Here duality gives an equivalence relation between different regions of the moduli space of the same theory. In this case, duality transformations form a symmetry group that acts on the moduli space of the theory. For example, type IIB is conjectured to have an $SL(2, Z)$ self-duality group.
- T-duality: In this case duality transformation maps the weak coupling region of one theory to the weak coupling region of another theory or the same theory. T-duality can be checked order by order in perturbation expansions[1].

# 3 Testing duality conjectures

In order to prove / test S duality we must be able to analyze at least one of the theories at strong coupling. Thus it seems impossible to prove or test any duality conjecture in string theory. This is where supersymmetry(SUSY) comes to our rescue. SUSY gives rise to certain non-renormalization theorems in string theory, due to which some of the weak coupling calculation can be trusted even at strong coupling[2].Thus we can focus our attention on such "non-renormalized" quantities and ask if they are invariant under the proposed duality transformations. Testing duality invariance of those quantities provides us with various tests of duality conjectures, and is in fact the basis of all duality conjectures. For theories with 16 or more SUSY generators the non-renormalization theorems are particularly powerful. In particular,

- Form of the low energy effective action involving the massless states of the theory is completely fixed by the requirement of SUSY (and the spectrum)[3]. Thus this effective action cannot get renormalized by string loop corrections. As a result, any valid symmetry of the theory must be a symmetry of this effective field theory. Actually, since the low energy effective action is to be used only for deriving the equations of motion from this action, and / or computing the tree level S-matrix elements using this action, but not to perform a full fledged path integral, it is enough that only the equations of motion derived from this action are invariant under duality transformations.

- These theories contain special class of states which are invariant under part of SUSY transformation. These BPS states are characterized by two properties:

  First, they belong to a supermultiplet which has typically less dimensions than a non-BPS state. This has an analog in the theory of representations of the Lorentz group, where massless states form a shorter representation of the algebra than massive states.

  Second, the mass of a BPS state is determined by its charge as a consequence of the SUSY algebra. This relation between the mass and the charge is known as the BPS mass formula. This statement also has an analog in the theory of representations of the Lorentz algebra, *e.g.* a spin 1 representation of the Lorentz algebra containing only two states must be necessarily massless.

Let me explain the origin of the two properties of the BPS states. Suppose the theory has $N$ real SUSY generators $Q_\alpha$ $(1 \le \alpha \le N)$. Acting on a single particle state at rest, the supersymmetry algebra takes the form:

$$\{Q_\alpha, Q_\beta\} = f_{\alpha\beta}(m, \overrightarrow{Q}, \{y\}) \tag{1}$$

where $f_{\alpha\beta}$ is a real symmetric matrix which is a function of its arguments $m, \overrightarrow{Q}$ and $\{y\}$. Here $m$ is the rest mass of the particle, $\overrightarrow{Q}$ denotes various gauge charges carried by the particle, and $\{y\}$ denotes the coordinates labelling the moduli space of the theory. Consider the following distinct cases:

1. $f_{\alpha\beta}$ has no zero eigenvalue. In this case by taking appropriate linear combinations of $Q_\alpha$ we can diagonalize $f$. By a further appropriate rescaling of $Q_\alpha$, we can bring $f$ into the identity matrix. Thus in this

basis the supersymmetry algebra has the form:

$$\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta} \tag{2}$$

This is the $N$ dimensional Clifford algebra. Thus the single particle states under consideration form a representation of this Clifford algebra, which is $2^{N/2}$ dimensional. ( We are considering the case where $N$ is even.) Such states would correspond to non-BPS states, known as *long* multiplets.

2. $f$ has $(N-M)$ zero eigenvalues for some $M < N$. In this case, by taking linear combinations of the $Q_\alpha$ we can bring the algebra into the form:

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= \delta_{\alpha\beta}, \text{ for } 1 \leq \alpha, \beta \leq M \\ &= 0 \text{ for } \alpha \text{ or } \beta > M \end{aligned} \tag{3}$$

We can form an irreducible representation of this algebra by taking all states to be annihilated by $Q_\alpha$ for $\alpha > M$. In that case the states will form a representation of an $M$ dimensional Clifford algebra generated by $Q_\alpha$ for $1 \leq \alpha \leq M$ . This representation is $2^{M/2}$ dimensional for $M$ even, also called *short* multiplets. When $M = N/2$, it's called *ultrashort*. Since $M < N$ , we see that there are lower dimensional representations compared to that of a generic non-BPS state. Furthermore, these states are invariant under part of the supersymmetry algebra generated by $Q_\alpha$ for $\alpha > M$. These are known as BPS states. We can get different kinds of BPS states depending on the value of $M$ , i.e. depending on the number of supersymmetry generators that leave the state invariant.

From this discussion it is clear that in order to get a BPS state, the matrix $f$ must have some zero eigenvalues. This in turn, gives a constraint involving mass $m$ , charges $Q$ and the moduli $\{y\}$, and is the origin of the BPS formula relating the mass and the charge of the particle.

BPS states are further characterized by the property that the degeneracy of BPS states with a given set of charge quantum numbers is independent of the value of the moduli fields $\{y\}$[4]. Since string coupling is also one of the moduli of the theory, this implies that the degeneracy at any value of the string coupling is the same as that at weak coupling. This is the key property of the BPS states that makes them so useful in testing duality, so let us review the argument leading to this property. Suppose the theory has an ultra-short multiplet at some point in the moduli space. Now let us change the moduli. The question that we shall be asking is: can the ultra-short multiplet become a long (or any other) multiplet as we change the moduli? If we assume that the total number of states does not change discontinuously, then this is clearly not possible since other multiplets have different number of states. Thus as long as the spectrum varies smoothly with the moduli (which we shall assume), an ultra-short multiplet stays ultra-short as we move in the moduli space. Furthermore, as long as it stays ultra-short, its mass is determined by the BPS formula. Thus we see that the degeneracy of ultra-short multiplets cannot change as we change the moduli of the theory. A similar argument can be given for other multiplets as well. Note that for this argument to be strictly valid, we require that the mass of the BPS state should stay away from the continuum, since otherwise the counting of states is not a well defined procedure. This requires that the mass of a BPS state should be strictly less than the total mass of any set of two or more particles carrying the same total charge as the BPS state.

Given this result, we can now adapt the following strategy to carry out tests of various duality conjectures using the spectrum of BPS states in the theory:

1. Identify BPS states in the spectrum of elementary string states. The spectrum of these BPS states can be trusted at all values of the coupling even though it is calculated at weak coupling.

2. Make a conjectured duality transformation. This typically takes a BPS state in the spectrum of elementary string states to another BPS state, but with quantum numbers that are not present in the spectrum of elementary string states. Thus these states must arise as solitons /composite states.

3. Try to explicitly verify the existence of these solitonic states with degeneracy as predicted by duality. This will provide a non-trivial test of the corresponding duality conjecture.

# 4 Examples of duality and notion of M-theory

## 4.1 General idea

For this part, we will resort to the evidence in the low energy effective actions (the corresponding supergravity actions) and the BPS states. I will follow 12.1 of Ref[5] and Ref[4] for the low energy effective actions. As to type IIB, the review and the corrected paper of Schwarz[6][7] are well readable and full of insight.

Before we go into details, let's first have a general idea of what will happen. The five superstring theories can be related to one another by a rich variety of dualities. The dualities that require compactification of only one spatial dimension, leaving a 9d Minkowski space-time, are sufficient to show that all five are related to one another. This strongly suggests that they are best regarded as different descriptions of a single underlying theory. Each one is better suited to describing some portion of the moduli space of possible vacua than the others.

Two of the relevant dualities are T dualities. When the IIA and IIB theories are each compactified on a circle, so that altogether the space-time topology is $R^9 \times S^1$, the two theories are T dual. This means that they describe identical physics, provided that the radius of one circle is the inverse of the other one (in string units). Suppose we compactify the $X^9$ coordinate in either theory and take the $R \longrightarrow 0$ limit. This is equivalent to the $R \longrightarrow \infty$ limit in the dual coordinate, whose right-moving part is reflected

$$X'^9_R(\bar{z}) = -X^9_R(\bar{z}) \tag{4}$$

By superconformal invariance we must also reflect $\tilde{\psi}'^9(\bar{z})$

$$\tilde{\psi}^9(\bar{z}) = -\tilde{\psi}^9(\bar{z}) \tag{5}$$

However, this means the chirality of the right-moving R sector ground state is reversed: the raising and lowering operators $\tilde{\psi}^8_0 \pm i\tilde{\psi}^9_0$ are interchanged. Simply put, T-duality is a spacetime parity operation on just one side of the world-sheet, and so reverses the relative chiralities of the right- and left-moving ground states. If we begin with the IIA theory and take the compactification radius to be small, we obtain the IIB at large radius, and vice versa. The same is true if we T-dualize on any odd number of dimensions, while T-dualizing on an even number of dimensions returns one to the type II theory one began. Actually T-duality also matches the R-R field strengths and potentials in IIA and IIB (see 13.1 of Ref[5]).

In a similar manner, one can show that the heterotic SO (32) and $E_8 \times E_8$ string theories are T dual when each of them is compactified on a circle (in this case, one needs Wilson lines. See 11.6 of Ref[5]).

In either of the above two cases, the noncompact theories are different limits of a single space of compactified theories.

The pair of T dualities described above provides two connections among the five superstring theories. To see that all five are connected requires examining non-perturbative dualities. One way of addressing the problem is to ask, for each of the five theories, whether the strong coupling limit has a dual weakly coupled description. The procedure is to identify a plausible candidate for an S dual description and then to examine its consequences. The result is that an interesting and consistent story emerges. In fact, it is so compelling that there can be little doubt about the truth of the proposed dualities. So let me now say what they are.

The type I and heterotic SO (32) string theories are S dual.[8][9] This means that the respective dilatons are related by $\phi_I = -\phi_H$, so that the coupling constants ( $e^{<\phi>}$ ) are reciprocal to one another. This identification is supported by the fact that both have the same low-energy effective field theory descriptions ( $N = 1$ supergravity coupled to SO (32) super Yang–Mills in 10d. See 12.1 of Ref[5]). The field redefinition $\phi_I = -\phi_H$ must be accompanied by a Weyl rescaling of the metric to convert from one version of the action to the other.

The strong coupling limits of the IIA and $E_8 \times E_8$ theories turn out to provide quite a different surprise. In each case there is an eleventh dimension (tenth spatial dimension) that becomes large at strong coupling.[6] Specifically, the size of this dimension scales as $L_{11} \sim (e^{\langle \phi^{(10)} \rangle})^{2/3}$. Such a compact dimension is completely invisible in perturbation theory (an expansion about $e^{\langle \phi^{(10)} \rangle} = 0$), which is why it passed unnoticed for so many years. In the IIA case, the hidden dimension is a circle $S^1$, whereas in the $E_8 \times E_8$ case it is a line interval $I$, or (more precisely) an $S^1/Z_2$ orbifold. This means that in the $E_8 \times E_8$ case one can visualize the space-time as an 11d space-time with two 10d faces, which are sometimes referred to as "end-of-the-world 9-branes," since they have nine spatial dimensions. One of the $E_8$ gauge groups is associated to each face. In any case, at strong coupling the faces move apart

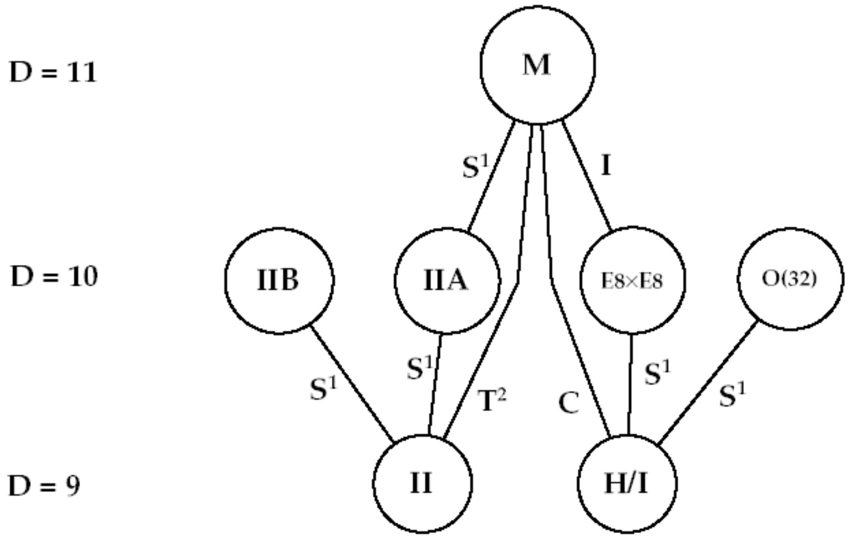


Figure 1: Duality connections

and (away from the faces) the theory is described by the same 11d bulk theory that describes the IIA theory at strong coupling. This 11d theory is described in leading order in a low-energy expansion by 11d supergravity, It is not yet known what is the correct algorithm that determines all the higher-dimension terms of the low-energy expansion of the effective action, but since we are confident that there is a consistent quantum theory, such an algorithm should exist. The unknown 11d quantum theory is referred to as M-theory. As we will discuss, certain of its supersymmetric solitons are known, and they provide a handle on many of its interesting properties.

The equivalences discussed above can be summarized by the diagram in Figure 1. This diagram is sufficient to show that the five superstring theories are all part of a single structure, but it is by no means the whole story. There are a variety of other surprising dualities that are only revealed upon compactification of additional dimensions. Some of these are discussed in [6].

The IIA/IIB T duality and the IIA/M S duality can be combined as a duality between IIB theory on $R^9 \times S^1$ and M-theory on $R^9 \times T^2$. This viewpoint turns out to be very powerful for understanding the structure of both the IIB and M theories separately, as will be discussed in considerable detail. The $T^2$ is characterized by three real parameters – its area $A_M$ and its modular parameter $\tau$, which characterizes its complex structure up to an $SL(2, Z)$ transformation. Indeed, we will find that in 9d the $SL(2, Z)$ modular group of the torus precisely corresponds to the $SL(2, Z)$ S duality group of the IIB theory. This geometrization of S duality is quite profound.

There is an analogous duality relating M-theory and the SO (32) theory (both type I and heterotic). This duality can be understood as arising as a corollary of the first one after modding out by a suitable $Z_2$ symmetry. In this case, M-theory compactified on a cylinder $C = I \times S^1$ is dual to the SO (32) theory compactified on a circle $S^1$.

## 4.2 IIA/M duality and definition of M-theory

I'd better first clarify the difference between string metric and canonical Einstein metric. The string metric $G_{\mu\nu}$ – the metric that is used in computing the area of the string world-sheet embedded in space time for calculating string scattering amplitudes. For a string theory compactified on a $(9-d)$ dimensional manifold $\mathcal{M}$ , we shall denote by $\phi^{(d+1)}$ the shifted dilaton (the one after compactification), related to the dilaton $\phi^{(10)}$ of the ten dimensional string theory as

$$2\phi^{(d+1)} = 2\phi^{(10)} - \ln V \tag{6}$$

where $(2\pi)^{9-d}V$ is the volume of $\mathcal{M}$ measured in the ten dimensional string metric. The dilaton is normalized in such a way that $e^{2\langle\phi^{(10)}\rangle}$ corresponds to the square of the closed string coupling constant in ten dimensions. $g_{\mu\nu}$ will denote the canonical Einstein metric which is related to the string metric by an appropriate conformal rescaling involving the dilaton field,

$$g_{\mu\nu} = e^{-\frac{4}{d-1}\phi^{(d+1)}} G_{\mu\nu} \tag{7}$$

It is well known that type IIA supergravity in 10d can be understood as arising from compactification of 11d supergravity on a circle (see 12.1 of Ref[5]). Doing this, the 11d metric (the canonical Einstein metric) gives rise to the 10d metric, a one-form $A$, and a dilaton $\phi_A$. Specifically, if $G^{(10)}$ denotes the IIA string metric,

$$g^{(11)}_{MN}dx^M dx^N = e^{-2\phi_A/3}G^{(10)}_{\mu\nu}dx^\mu dx^\nu + e^{4\phi_A/3}(dx^{11} - A_\mu dx^\mu)^2 \tag{8}$$

Identifying the string coupling constant $\lambda = e^{<\phi_A>}$, one sees that $R_{11} \sim \lambda^{2/3}$,

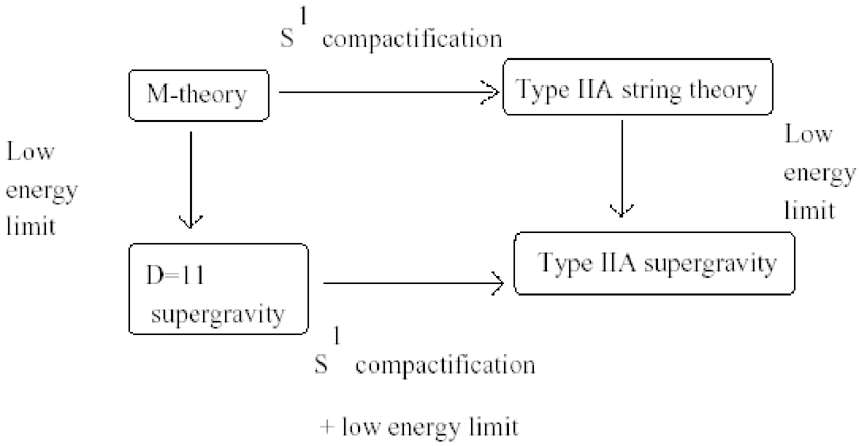


Figure 2: The relationship between M-theory and Type IIA supergravity / string theories.

as was asserted earlier. Also, the 11d three-form $C$ decomposes in 10d into a three-form $C$ and a two-form $B$.

This leads one to the conjecture[10] that *in the strong coupling limit type IIA string theory approaches an 11 dimensional Lorentz invariant theory, whose low energy limit is 11-dimensional N=1 supergravity.* This theory has been called M-theory. The situation is illustrated in Figure 2. Part of the conjecture is just the definition of M-theory as the strong coupling limit of type IIA string theory. The non-trivial part of the conjecture is that it describes a Lorentz invariant theory in eleven dimensions.

The evidence for the existence of an eleven dimensional theory also involves precise tests of the spectrum of BPS states. M-theory on $S^1$ will have Kaluza-Klein modes representing states in the eleven dimensional N=1 supergravity multiplet carrying momentum along the compact $x^{10}$ direction. These are BPS states, and can be shown to belong to the 256 dimensional ultra-short representation of the supersymmetry algebra. (a similar example in Heterotic superstring theories is given in P79-81 of Ref[5].) The charge quantum number characterizing such a state is the momentum ($k/R$) along $S^1$. Thus for every integer $k$ we should find such BPS states in type IIA string theory in ten dimensions. In M-theory these states carry $k$ units of $g^{(S)}_{10\mu}$ ($g^{(S)}_{MN}$ is the canonical Einstein metric in the 11d supergravity) gets mapped to $A_\mu$ under the IIA/M-theory duality, these states must carry $k$ units of $A_\mu$ charge in type IIA string theory. If we now recall that in type IIA string theory $A_\mu$ arises in the R-R sector, we see that these states cannot come from elementary string states, as elementary string excitations are neutral under R-R sector gauge fields (see P88 of Ref[5]). However, Dirichlet 0-branes in this theory do carry $A_\mu$ charge. In particular the state with $k = 1$ corresponds to a single Dirichlet zero brane. As usual, the collective coordinate dynamics of the 0-branes is determined from the dynamics of massless open string states with ends lying on the D0-brane, and in this case is described by the dimensional reduction of N=1 super-Maxwell (U(1) gauge) theory (N≠2 because D$p$-branes are invariant under half of the space-time supersymmetry generators. See P55 of Ref[4] or P139-140 of Ref[5].) from $(9+1)$ to $(0+1)$ dimensions (the bosonic gauge field becomes the collective coordinates under dimensional reduction). This theory has sixteen fermion zero modes whose quantization leads to a $2^8 = 256$-fold degeneracy for a quantum state labelled by the momenta conjugate to the collective coordinates. Thus we see that we indeed have an ultra-short multiplet with unit $A_\mu$ charge, as predicted by the IIA/M-theory duality conjecture.

For $k > 1$,these must arise in type IIA string theory as bound states of $k$ D0-branes. Dynamics of collective coordinates of $k$ D0-branes is given by the dimensional reduction of N=1 supersymmetric U($k$) gauge theory from $(9+1)$ to $(0+1)$ dimensions. Thus the number of ultra-short multiplets with $k$-units of $A_\mu$ charge is determined in terms of the number of normalizable supersymmetric ground states of this quantum mechanical system. Finding these bound states is much more difficult than the bound state problems discussed earlier. The main obstacle to this analysis is that a charge $k$ state has the same energy as $k$ charge 1 states at rest. Thus the bound states we are looking for sit at the bottom of a continuum. Such states are difficult to study. For $k = 2$ such a bound state with the correct degeneracy has been found[11].

## 4.3 IIB $SL(2, Z)$ self-duality and global diffeomorphism of M-theory

### 4.3.1 Constructing the solution

As has already been noted, the type IIB superstring in 10d has two two-form potentials, $B^{(1)}_{\mu\nu}$(NS) and $B^{(2)}_{\mu\nu}$(R-R). Therefore, string-like solutions can, in general, carry a pair of charges

$$q_I \tilde{} \int_{S^7} *dB^{(I)}, \quad I = 1, 2 \tag{9}$$

Let us construct these solutions explicitly. To make the $SL(2,R)$ symmetry of the supergravity field equations manifest, it is convenient to introduce a two-component vector notation

$$H = dB = \begin{pmatrix} dB^{(1)} \\ dB^{(2)} \end{pmatrix} \tag{10}$$

It is also convenient to combine the R-R scalar $\chi$ and the dilaton $\phi$ into a complex scalar field

$$\rho = \chi + ie^{-\phi} \tag{11}$$

and to represent the vev of this field by

$$\langle\rho\rangle = \rho_0 = \chi_0 + ie^{-\phi_0} = \frac{\theta}{2\pi} + \frac{i}{\lambda_B} \tag{12}$$

In the low energy IIB supergravity, equations of motions are covariant (they transform into each other) under $SL(2,R)$ symmetry[7][4]:

$$\rho \longrightarrow \frac{a\rho + b}{c\lambda + d}, \quad \begin{pmatrix} B^{(1)}_{\mu\nu} \\ B^{(2)}_{\mu\nu} \end{pmatrix} \longrightarrow \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B^{(1)}_{\mu\nu} \\ B^{(2)}_{\mu\nu} \end{pmatrix}, \quad ad - bc = 1 \tag{13}$$

with $a, b, c, d$ real numbers.

The exact symmetry is not $SL(2,R)$, but $SL(2,Z)$. The breaking of $SL(2,R)$ to $SL(2,Z)$ can be seen as follows. An elementary string is known to carry $B^{(1)}_{\mu\nu}$ charge. In suitable normalization convention, it carries exactly one unit of $B^{(1)}_{\mu\nu}$ charge. This means that the $B^{(1)}_{\mu\nu}$ charge must be quantized in integer units, as the spectrum of string theory does not contain fractional strings carrying a fraction of the charge carried by the elementary string. From Eq.(13), we see that acting on an elementary string state carrying one unit of $B^{(1)}_{\mu\nu}$ charge, the $SL(2,R)$ transformation gives a state with $d$ units of $B^{(1)}_{\mu\nu}$ charge and $-b$ units of $B^{(2)}_{\mu\nu}$ charge. Thus $d$ must be an integer. It is easy to see that the maximal subgroup of $SL(2,R)$ for which $d$ is always an integer consists of matrices of the form

$$\begin{pmatrix} a & \alpha b \\ \alpha^{-1}c & d \end{pmatrix} \tag{14}$$

with $a, b, c, d$ integers satisfying $ad - bc = 1$, and $\alpha$ a fixed constant. Absorbing $\alpha$ into a redefinition of $B^{(2)}_{\mu\nu}$ (old$B^{(2)}_{\mu\nu} = (\alpha^{-1})$new$B^{(2)}_{\mu\nu}$), we get $SL(2, Z)$.

It is also convenient to introduce the symmetric $SL(2, R)$ matrix

$$M = e^{\phi}\begin{pmatrix} |\rho|^2 & \chi \\ \chi & 1 \end{pmatrix} \tag{15}$$

which transforms by the simple rule,

$$M \longrightarrow \Lambda M \Lambda^T \tag{16}$$

The $B$ fields transform linearly by the rule $B \longrightarrow (\Lambda^T)^{-1}B$, while the canonical Einstein metric $g_{\mu\nu}$ and the 4-form $A_4$ are invariant. Note that since the dilaton transforms, the IIB string metric

$$G^{(B)}_{\mu\nu} = e^{\phi/2}g_{\mu\nu} \tag{17}$$

is not $SL(2, R)$ invariant.

The string-like solutions of the IIB supergravity field equations that we are seeking have $A_4$ and all fermi fields equal to zero. The reason for setting $A_4$ to zero is that the corresponding charges are carried by a self-dual three-brane, whereas the focus here is on charges carried by strings. While it is difficult to formulate a convenient action that gives the complete field equations (because the field strength of $A_4$ is self-dual), it is not hard to find the action that gives the field equations with $A_4$ and the fermi fields set equal to zero. It is

$$S = \int d^{10}x\sqrt{-g}(R - \frac{1}{12}H^T_{\mu\nu\rho}MH^{\mu\nu\rho} + \frac{1}{4}tr(\partial^\mu M\partial_\mu M^{-1})) \tag{18}$$

This action is manifestly invariant under global $SL(2, R)$ transformations. The solution we seek consists of a string-like soliton along the $x^1$ axis with $B$ charges $(q_1, q_2)$ and vacuum defined by $\rho(r) \sim \rho_0$ as $r \longrightarrow \infty$, where $r^2 = \overrightarrow{x} \cdot \overrightarrow{x}$, and $\overrightarrow{x}$ refers to the eight transverse directions $x^2, x^3, ..., x^9$. Also, the metric should approach the Minkowski metric as $r \longrightarrow \infty$.

Our problem was solved some time ago, for a special case, namely $\overrightarrow{q}$ $= (1, 0)$ and $\rho_0 = i$. All ten-dimensional supergravity theories contain the following terms in common

$$S_0 = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-\phi}H^2) \tag{19}$$

where $H = dB$ is the three-form field strength and $\phi$ is the dilaton. Moreover, in each case, a solution to the classical equations of motion derived from $S_0$ can be regarded as a solution of the complete supergravity theory with all other fields set equal to zero. A macroscopic string-like solution, which was identified with the heterotic string, was constructed by Dabholkar *et al.[12]*. For Type IIB, the solution corresponds to $\chi = 0$, $H^{(2)} = 0$. The solution, restricted to $D = 10$, is given by

$$ds^2 = A^{-3/4}[-dt^2 + (dx^1)^2] + A^{1/4}d\mathbf{x}\cdot d\mathbf{x} \tag{20}$$

$$B_{01} = e^{2\phi} = A^{-1} \tag{21}$$

where

$$A = 1 + \frac{Q}{3r^6} \tag{22}$$

$\mathbf{x} = (x^2, x^3, ..., x^9)$, $\mathbf{x}\cdot\mathbf{x} = r^2 = \delta_{ij}x^i x^j$, and $Q$ is the $B_{\mu\nu}$ electric charge carried by the string. (Recall that the electric charge of a $(p + 2)$-form field strength is carried by a $p$-brane.) The expression for $Q$ is determined from the coefficient of $\nabla^2 A$ delta-function singularity at $r = 0$[7] to be

$$Q = \kappa^2 T_1^{(B)}/\omega_7 \tag{23}$$

where $\omega_7 = \frac{1}{3}\pi^4$ is the volume of $S^7$, $T_1^{(B)}$ is the fundamental string tension.

In type IIB, there are two three-form field strengths. Given the $SL(2, R)$ symmetry, it is clearly artificial to only consider solutions carrying $H^{(1)}$ electric charge and not $H^{(2)}$ electric charge. Measured in units of $Q$, we will consider solutions carrying charges $(q_1, q_2)$. Since there exist five-brane solutions carrying magnetic $H$ charge, the generalized Dirac quantization condition[13] implies that $q_1$ and$q_2$ must be integers. Moreover, $q_1$ and $q_2$ should be relatively prime, since otherwise the solution is neutrally stable against decomposing into a multiple string solution — the number of strings being given by the common divisor. Also, the $(q_1, q_2)$ string and the $(-q_1, -q_2)$ string are related by orientation reversal ($x^1 \longrightarrow -x^1$). A complete description of string solutions requires specifying the vacuum in which they reside. In the IIB theory this means choosing the asymptotic value of $\rho$ as $r \longrightarrow \infty$. The choice that is simplest, is $\rho_0 = i$, which corresponds to $\chi_0 = \phi_0 = 0$. The tension of the $(q_1, q_2)$ string, denoted by $T_q$, will be determined.

Replacing $T$ by $T_q$ in the solution described above still gives a classical solution (for coupling a string of tension $T_q$ rather than $T$ ), though in general it violates the quantization condition. To describe this solution, we replace $A(r)$ by $A_q(r)$, where

$$A_q(r) = 1 + \frac{\alpha_{(q_1,q_2)}}{3r^6} \tag{24}$$

Now consider letting $\alpha_{(q_1,q_2)} = \sqrt{q_1^2 + q_2^2}Q$ and $T_q = \sqrt{q_1^2 + q_2^2}T_1^{(B)}$. For this choice, we can recover the quantization condition by applying an $SL(2,R)$ transformation given by

$$\Lambda = \frac{1}{\sqrt{q_1^2 + q_2^2}} \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix} \tag{25}$$

to obtain a new classical solution. We have chosen $\Lambda$ to belong to the $SO(2)$ subgroup of $SL(2,R)$, because this preserves the modulus $\rho_0 = i$. In this way we obtain a solution with charges $(q_1, q_2)$ given by

$$ds^2 = A_q^{-3/4}[-dt^2 + (dx^1)^2] + A_q^{1/4} d\mathbf{x} \cdot d\mathbf{x} \tag{26}$$

$$B_{01}^{(i)} = \frac{q_i}{\sqrt{q_1^2 + q_2^2}} A_q^{-1} \tag{27}$$

$$\rho = \frac{iq_1 A_q^{1/2} - q_2}{iq_2 A_q^{1/2} + q_1} = \frac{q_1 q_2 (A_q - 1) + i(q_1^2 + q_2^2) A_q^{1/2}}{q_1^2 + q_2^2 A_q} \tag{28}$$

It is evident that as $r \longrightarrow \infty$, $A_q \longrightarrow 1$, and therefore $\rho \longrightarrow i$.

The next step is to generalize this family of solutions to an arbitrary vacuum modulus $\rho_0$. For this purpose we start with arbitrary $\alpha_{(q_1,q_2)}$, $\rho_0 = i$, $B^{(1)}{=}\cos\theta A_q^{-1}$, $B^{(2)}{=}\sin\theta A_q^{-1}$ and $\rho = (i\cos\theta A_q^{1/2} - \sin\theta)/(i\sin\theta A_q^{1/2} + \cos\theta)$. (The solution in Eq.(28) corresponds to the choice $e^{i\theta} = (q_1 + iq_2)/\sqrt{q_1^2 + q_2^2}$.) Next we apply $SL(2,R)$ transformation

$$\Lambda = \begin{pmatrix} e^{-\phi_0/2} & \chi_0 e^{\phi_0/2} \\ 0 & e^{\phi_0/2} \end{pmatrix} \tag{29}$$

which maps $i \longrightarrow \rho_0$. Finally, we must satisfy the quantization condition. This is achieved by the choice

$$e^{i\theta} = e^{\phi_0/2}(q_1 - q_2\rho_0)\Delta^{-1/2}_{(q_1,q_2)} \tag{30}$$

$$\Delta_{(q_1,q_2)} = (q_1 \quad q_2)\, M_0 \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = e^{\phi_0}(q_2\chi_0 - q_1)^2 + e^{-\phi_0}q_2^2 \tag{31}$$

$$= e^{\phi_0}\,|q_1 - q_2\rho_0|^2 \tag{32}$$

which gives the $SL(2,Z)$ covariant result

$$\alpha_{(q_1,q_2)} = \Delta^{1/2}_{(q_1,q_2)} Q \tag{33}$$

The general solution describing a $(q_1, q_2)$ string in the $\rho_0$ vacuum now follows. It is given by the metric in Eq.(26) and

$$B^{(i)}_{01} = (M_0^{-1})_{ij} q_j \Delta^{-1/2}_{(q_1,q_2)} A_q^{-1} \tag{34}$$

$$\rho = \frac{q_1\chi_0 - q_2\,|\rho_0|^2 + iq_1 e^{-\phi_0} A_q^{1/2}}{q_1 - q_2\chi_0 + iq_2 e^{-\phi_0} A_q^{1/2}} \tag{35}$$

which satisfies $\rho \longrightarrow \rho_0$.

We have now obtained an $SL(2,Z)$ family of type IIB macroscopic strings carrying $H$ charges $(q_1, q_2)$ and having tensions

$$T_{(q_1,q_2)} = \Delta^{1/2}_{(q_1,q_2)} T_1^{(B)} \tag{36}$$

To get a sense of the meaning of this equation, it is convenient to restrict to the special case $\chi_0 = 0$, so that $\rho_0 = i/\lambda_B$. Then the tension of the $(q_1, q_2)$ string in the canonical metric is

$$T_{(q_1,q_2)} = (\lambda_B q_1^2 + \lambda_B^{-1} q_2^2)^{1/2} T_1^{(B)} \tag{37}$$

Converting to the IIB string metric, redefines this by a factor of $\lambda_B^{-1/2}$, giving

$$\tilde{T}_{(q_1,q_2)} = (q_1^2 + \lambda_B^{-2} q_2^2)^{1/2} T_1^{(B)} \tag{38}$$

Thus, in the string metric, the fundamental string tension is a constant, $\tilde{T}_{(1,0)} = T_1^{(B)}$. The D-string, which carries R-R charge only, on the other hand has tension $\tilde{T}_{(0,1)} = \lambda_B^{-1} T_1^{(B)}$. The scaling $T \sim \lambda^{-1}$ is characteristic D-branes in the string metric.

As is typical of BPS mass formulas, the tensions we have found satisfy a triangle inequality

$$T_{(q_1+p_1,q_2+p_2)} \leq T_{(q_1,q_2)} + T_{(p_1,p_2)} \tag{39}$$

and equality requires that $\vec{p}$ and $\vec{q}$ are parallel. This means that if $q_1$ and $q_2$ are relatively prime, a string with charges $(q_1, q_2)$ and tension $T_{(q_1,q_2)}$ is absolutely stable, protected by charge conservation and a "tension gap" (the analog of a mass gap) from decay into multiple strings. On the other hand, a string with charges $(nq_1, nq_2)$ is at the threshold for decay into $n$ $(q_1, q_2)$ strings. Whether one has a bound state or not, in such a case, is a delicate issue whose answer depends on the particular problem. We will show that the duality relation to M-theory requires that only strings with $q_1$ and $q_2$ relatively prime be included. This conclusion is supported by a bound-state analysis carried out by Witten.[14] One way of stating the conclusion is that $q_1$ fundamental strings and $q_2$ D -strings (all of which are parallel) can form a single bound state if and only if $q_1$ and $q_2$ are relatively prime.

### 4.3.2 Compactification and comparison

Let us now consider type IIB string theory compactified on a circle of radius $R_B$ (and circumference $L_B = 2\pi R_B$). Since all of the $(q_1, q_2)$ strings are related by $SL(2, Z)$ transformations, they are all equivalent and any one of them can be weakly coupled. However, when one is weakly coupled, all the others are necessarily strongly coupled. Nevertheless, let us consider an arbitrary $(q_1, q_2)$ string and write down the spectrum of its 9d excitations in the limit of weak coupling. This is given by standard string theory formulas:

$$\begin{aligned} m_B^2 &= \left(\frac{n}{R_B}\right)^2 + \left(2\pi R_B w T_{(q_1,q_2)}\right)^2 + 4\pi T_{(q_1,q_2)}(N_L + N_R) && (40) \\ &= \left(\frac{n}{R_B} + 2\pi R_B w T_{(q_1,q_2)}\right)^2 + 8\pi N_L T_{(q_1,q_2)} && (41) \\ &= \left(\frac{n}{R_B} - 2\pi R_B w T_{(q_1,q_2)}\right)^2 + 8\pi N_R T_{(q_1,q_2)} && (42) \end{aligned}$$

Here $n$ is the Kaluza–Klein excitation number and $w$ is the string winding number. $N_L$ and $N_R$ are excitation numbers of left-moving and right-moving oscillator modes, and the level-matching condition is

$$N_R - N_L = nw \tag{43}$$

Now our purpose is to use this formula for all the $(q_1, q_2)$ strings simultaneously. However, the formula is completely meaningless at strong coupling, and (as we have said) at most one of the strings is weakly coupled. The appropriate trick in this case is to consider only BPS states – ones belonging to short supersymmetry multiplets. They are easy to identify, being given by either $N_R = 0$ or $N_L = 0$. (Ones with $N_L = N_R = 0$ are ultrashort.)[12] $N_L = 0$ (or $N_R = 0$) will make the state invariant under $1/4$ of the real spacetime SUSY generators which are $1/2$ of the real spacetime SUSY generators coming from the left-(or right-) moving sector of the worldsheet. For these states the mass formula should be exact, even at strong coupling. Therefore, it can be used for all the strings at the same time. In this way, we obtain reliable mass formulas for a very large part of the spectrum – much more than appears in perturbation theory. Of course, the appearance of this rich spectrum of BPS states depends crucially on the compactification. Using Eqs.(32)(37), the winding-mode term in Eq.(40) contains the factor

$$(wT_{(q_1,q_2)})^2 = w^2 \Delta_{(q_1,q_2)} (T_1^{(B)})^2 = e^{\phi_0} \left| l_1 - l_2 \rho_0 \right|^2 (T_1^{(B)})^2 \tag{44}$$

where $(l_1, l_2) = w\,(q_1, q_2)$. There is a unique correspondence between the three integers $w, q_1, q_2$ and an arbitrary pair of integers $l_1, l_2$. The integer $w$ is the greatest common divisor of $l_1$ and $l_2$. The only ambiguity is whether to choose $w$ or $-w$, but since $w$ is (oriented) winding number and the $(-q_1, -q_2)$ string is the orientation-reversed $(q_1, q_2)$ string, the two choices are actually equivalent. Thus BPS states are characterized by three integers $n, l_1, l_2$ and oscillator excitations corresponding to $N_L = |nw|$, tensored with a 16-dimensional short multiplet from $N_R = 0$ sector (or vice versa).

Let us now consider 11d M-theory compactified on a torus. The torus is characterized by a complex modulus $\tau = \tau_1 + i\tau_2$ (as usual) and by its area $A_M$, measured in the 11d canonical Einstein metric. If the two periods of the torus are $2\pi R_{11}$ and $2\pi R_{11}\tau$, then $A_M = (2\pi R_{11})^2 \tau_2$. In terms of coordinates $z = x + iy$ on the torus, a single-valued wave function has the form

$$\phi_{l_1,l_2} \,\tilde{}\, \exp\{\frac{i}{R_{11}}[x l_2 - \frac{y}{\tau_2}(l_2\tau_1 - l_1)]\} \tag{45}$$

These characterize Kaluza–Klein excitations. The contribution to the mass-squared is given by the eigenvalue of $-\partial_x^2 - \partial_y^2$,

$$\frac{1}{R_{11}^2}(l_2^2 + \frac{1}{\tau_2^2}(l_2\tau_1 - l_1)^2) = \frac{|l_1 - l_2\tau|^2}{(\tau_2 R_{11})^2} \tag{46}$$

Clearly, this term has the right structure to match the string winding-mode terms described in Eq.(44). To go further, we also need an M-theory counterpart of the term $\left(\frac{n}{R_B}\right)^2$ in the IIB string mass formula. Here there is also a natural candidate: wrapping M-theory 2-branes so as to cover the torus $m$ times. If the 2-brane tension is $T_2^{(M)}$, this gives a contribution $\left(A_M T_2^{(M)} m\right)^2$ ($A_M = (2\pi R_{11})^2 \tau_2$ is the area of the torus) to the mass-squared. Thus

$$m_{11}^2 = \left(m\,(2\pi R_B)^2\,\tau_2 T_2^{(M)}\right)^2 + \frac{|l_1 - l_2\tau|^2}{(\tau_2 R_{11})^2} + ... \tag{47}$$

where the dots represent membrane excitations, which we do not know how to compute. This is to be compared to the Eqs.(40)(44) for $m_B^2$, allowing $m_{11} = \beta m_B$, since they are measured in different metrics. Agreement of the formulas is only possible if the vacuum modulus $\rho_0$ of the IIB theory is identified with the modular parameter $\tau$ of the torus.

$$\tau = \rho_0 \tag{48}$$

Since $SL(2,Z)$ is the modular group of the torus, this provides strong evidence that it should also be the self-duality group of the IIB string. In addition, the identification $m_{11} = \beta m_B$ gives

$$G^{(M)} = \beta^2 G^{(B)} \tag{49}$$
$$\beta^2 = 2\pi R_{11} e^{-\phi_0/2} T_2^{(M)}/T_1^{(B)} \tag{50}$$
$$R_B^{-2} = T_1^{(B)} T_2^{(M)} A_M^{3/2} \tag{51}$$

We still have the oscillator excitations of the type IIB string BPS mass formula to account for. Their M-theory counterparts must be excitations of the wrapped 2-brane. Unfortunately, since the quantization of the 2-brane is not yet understood, this cannot be checked. The story could be turned around at this point to infer what the BPS excitation spectrum of wrapped

2-branes must be. Maybe, trying to understand this spectrum will lead to a better understanding of 2-brane quantization. In any case, assuming that this works, we have found that Kaluza–Klein excitations of the type IIB theory compactified on a circle correspond to wrappings of the 2-brane on the torus and that Kaluza–Klein modes of M-theory on the torus correspond to windings of an infinite family of type IIB strings on the circle.

We have conjectured that M-theory compactified on a torus of area $A_M$ and modular parameter $\tau$ is identical to type IIB string theory compactified on a circle of circumference $L_B = 2\pi R_B$ and vacuum parameter $\rho_0$. The conjecture was supported by matching BPS 0-branes in 9d, which dictated how to match parameters ($\tau = \rho_0$ *etc.*). We can carry out additional tests of the proposed duality, and learn interesting new relations at the same time, by also matching BPS $p$-branes with $p > 0$ in 9d. Here we will describe the result for $p = 1$ (strings) and leave the other cases to Ref([6]).

Trivial reduction of the IIB strings in 10d gives strings with the same charges $(q_1, q_2)$ and tensions $T_{(q_1,q_2)}$ in 9d. The interesting question is how these should be interpreted in M-theory. The way to do this is to start with a 2-brane of toroidal topology in M-theory and to wrap one of its cycles on a $(q_1, q_2)$ homology cycle of the spatial torus. The minimal length of such a cycle is

$$L_{(q_1,q_2)} = 2\pi R_{11} \, |q_1 - q_2\tau| = \left(A_M \Delta_{(q_1,q_2)}\right)^{1/2} \tag{52}$$

Thus, this wrapping gives a 9d string whose tension is

$$T^{(11)}_{(q_1,q_2)} = L_{(q_1,q_2)} T_2^{(M)} \tag{53}$$

The superscript 11 emphasizes that this is measured in the 11d metric. To compare with the IIB string tensions, we use Eqs.(49)(50) to deduce that

$$T_{(q_1,q_2)} = \beta^{-2} T^{(11)}_{(q_1,q_2)} = \left(\Delta_{(q_1,q_2)}\right)^{1/2} T_1^{(B)} \tag{54}$$

This agrees with the result in Eq.(36), showing that this is a correct interpretation.

### 4.3.3 Definition of M-theory revisited

Recall that in IIA/M duality, we conjectured a definition of M-theory as the strong coupling limit type IIA. But we had no evidence why M-theory is an 11 dimensional Lorentz invariant theory. We now have the evidence:

- First of all, from Lorentz invariance of type IIA string theory we know that we have Lorentz invariance in coordinates $x_0, ...x_9$ when all the coordinates $x_0, ...x_9$ are non-compact.
- Then from the conjectured $SL(2, Z)$ duality symmetry of type IIB string theory we know that we have an exchange symmetry between the 9th and the 10th coordinate of M-theory when these coordinates are compact. In the limit when the radius of both the compact circles are taken to be large, this would mean that we should have Lorentz invariance in coordinates $x_0, ...x_{10}$.

## 4.4 The parallel branch–$\mathbf{E}_8\times\mathbf{E}_8$ and SO(32)

Let's look at Figure 1 again. The figure is pretty symmetric if we think $\mathrm{E}_8\times\mathrm{E}_8$ as the counterpart of IIA and SO(32) as the counterpart of IIB. It's very natural to think that there is a second duality that relates M-theory compactified on $S^1/Z_2 \times S^1$ to SO(32) theory compactified on $S^1$. Since $S^1/Z_2$ can be regarded as a line interval $I$, $S^1/Z_2 \times S^1$ can be regarded as a cylinder $C$. We will choose its height to be $L_1$ and its circumference to be $L_2 = 2\pi R_2$. The circumference of the circle on which the SO(32) theory is compactified is denoted $L_O = 2\pi R_O$ in the canonical 10d metric.

Before describing $p$-brane matching in 9d, let us briefly review the Horava–Witten picture of the $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string theory.[9] Compactification of M-theory on $S^1/Z_2 = I$ gives a space-time with two 10d faces, separated by a distance $L_1 \sim \lambda_H^{2/3}$, where $\lambda_H$ is the coupling constant of the $\mathrm{E}_8\times\mathrm{E}_8$ theory in 10d. The two 10d faces are sometimes called "end-of-the-world 9-branes." Each of them carries the gauge fields for one of the two $\mathrm{E}_8$'s. For reasons that are explained in Ref.[6], M-theory 2-branes are allowed to terminate on a face, so that the boundary of the 2-brane is a circle inside the face. In this picture, an $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string is a cylindrical 2-brane suspended between the two faces, with one $\mathrm{E}_8$ current algebra associated to each boundary. This cylinder (or strip) is well approximated by a string living in 10d when the separation $L_1$ is small. Since perturbation theory in $\lambda_H$ is an expansion about $L_1 = 0$, the fact that there really are eleven dimensions and that the string is actually a membrane is invisible in that approach.

The story described above is very similar to the relation between the IIA superstring and M-theory. In that case the compact dimension is a circle and the IIA string arises from wrapping the M-theory 2-brane around the circle. Thus, in a sense, the non-perturbative $E_8 \times E_8$ theory just involves modding out the non-perturbative IIA theory by a $Z_2$. (However, the rules for carrying out the modding in M-theory, which is not a string theory, are not so obvious.) Similarly, the type I SO(32) string theory in 10d can be constructed as a $Z_2$ orientifold of the type IIB theory in 10d. Thus, the duality we are considering now can be viewed as arising from modding out the previous one by a Z2 on both sides. However, in the following we will treat it as a separate problem instead of attempting to exploit that picture. Because of the similarity of the two problems, fewer details will be provided this time.

The SO(32) theory in 10d has both a type I and a heterotic description, which are S dual. That is, their coupling constants, $\lambda = e^{<\phi>}$, satisfy $\lambda_H^{(0)} = \left(\lambda_I^{(0)}\right)^{-1}$. As before, we match supersymmetry-preserving (BPS) branes in 9d. Recall that in the SO(32) theory, there is just one two-form field $B_{\mu\nu}$, and the $p$-branes that couple to it are the SO(32) heterotic string and its magnetic dual, which is a 5-brane. The type I open and closed strings do not carry a conserved charge and are not BPS. This is the reason that they can break. So from the type I viewpoint it is clear that the heterotic string can give a 0-brane or a 1-brane in 9d and that the dual 5-brane can give a 5-brane or a 4-brane in 9d. In each case, the issue is simply whether or not one cycle wraps around the spatial circle.

Now we need to find the corresponding 9d $p$-branes from the M-theory viewpoint, to understand why they are the only ones, and to explore what can be learned from matching tensions. We described how the $E_8 \times E_8$ string arises in 10d from wrapping the M-theory 2-brane on $I$. Subsequent reduction on a circle can clearly give a 0-brane or a 1-brane. But why is there no BPS 2-brane in 9d? When the 2-brane is forced to be in a 10d boundary, rather than in the 11d bulk, it becomes breakable (non-BPS). The technical reason is that there is no 3-form gauge field on the boundary (or in the 10d reduction). The story for the five-brane is just the reverse. Whereas the 2-brane must wrap on the $I$ dimension, the five-brane must not do so. As a result it gives a 5-brane or a 4-brane in 9d according to whether or not it wraps around the $S^1$ dimension. So, altogether, both pictures give the electric-magnetic dual pairs $(0, 5)$ and $(1, 4)$ in 9d.

From the $p$-brane matching one learns that[9]

$$\lambda_H^{(0)} = \frac{L_1}{L_2} \tag{55}$$

Thus, the SO(32) heterotic string is weakly coupled when the spatial cylinder of the M-theory compactification is a thin ribbon ($L_1 \ll L_2$). This is consistent with the earlier conclusion that the $E_8 \times E_8$ heterotic string is weakly coupled when $L_1$ is small. Conversely, the type I string is weakly coupled for $L_2 \ll L_1$, in which case the spatial cylinder is long and thin. The $Z_2$ transformation that inverts the modulus of the cylinder, $L_1/L_2$, corresponds to the type I/heterotic duality of the SO(32) theory. Since it is not a symmetry of the cylinder it implies that two different-looking string theories are S dual. This is to be contrasted with the $SL(2, Z)$ modular group symmetry of the torus, which accounts for the self-duality of the IIB theory.

The $p$-brane matching in 9d gives the relation

$$L_1 L_2^2 T_2^{(M)} = \left( \frac{T_1^{(O)} L_O^2}{2\pi} \right)^{-1} \tag{56}$$

which is the analog of Eq.(51). As in that case, it tells us that for fixed modulus $L_1/L_2$, one has the scaling law $L_O \tilde{} A_C^{-3/4}$, where $A_C = L_1/L_2$ is the area of the cylinder.

# 5 Comments on duality and M-theory

The duality relations discussed so far can be shown to be interrelated. Sen[16][4] had some arguments that all conjectured dualities can be 'derived' from the duality between type I and SO(32) heterotic string theory, T-dualities, and the definition of M- and F- theories.

I left out F-theory, U duality or duality relations from compactification below 9d. However, the above discussion has shown us strong evidence that all of the different superstring theories are equivalent. Each of them represents a perturbation expansion of a single underlying theory about a distinct point in the moduli space of quantum vacua. Moreover, there is a sixth rather special point in this moduli space characterized by an 11d Poincaré-invariant vacuum. The rules for doing quantum mechanics in the 11d vacuum are not yet well understood, but the answer (whatever it is) has been named 'M-theory'. Each of these descriptions (extended by various possible compactifications) gives access to different 'patches' of the space of quantum vacua. A good analogy is that the underlying theory is being defined in much the same way that one defines a manifold. A manifold can be defined by giving a covering by open sets that are diffeomorphic to open sets in $\mathbf{R}^n$ and by consistently defining transition functions on their overlaps. In the proposed analogy, each of the superstring theories corresponds to one of the open sets, and the dualities that characterize their non-perturbative equivalences correspond to the transition functions. From this viewpoint the various dualities could be viewed as part of the *definition* of the underlying theory rather than as conjectured theorems that require proof. Of course, as in the case of ordinary transition functions, they must satisfy a number of consistency conditions. Exactly what consistency conditions are required has not yet been carefully spelled out. There is a widespread belief that a deeper formulation of the theory ought to exist, but it is also conceivable that the various perturbation expansions and non-perturbative dualities constitute the best definition of the theory.

Our definition of M-theory does not give us any clue as to how to systematically compute S-matrix elements in M-theory, since the type IIA string theory is defined only by the rules for its perturbation expansion in the coupling constant. Thus finding a non-perturbative definition of M-theory is of importance. One might try various approaches:

1. M-theory is the N=1 supergravity theory in D=11: This proposal by itself does not make sense since this supergravity theory is not renormalizable and hence is plagued by the usual ultra-violet divergences. One might argue that there is some intrinsic regularization that accompanies this supergravity theory, but till we find such a regularization, describing M-theory as the supergravity theory remains an empty statement.

2. M-theory is a string theory: One might imagine that M-theory can be made finite by regarding it as the low energy limit of some string theory in the same way that the various supergravity theories in ten dimensions are made finite. However, nobody has been able to find any such string theory so far. There is also another compelling reason to believe that M-theory is not a string theory. With the sole exception of type I string theory, every other string theory has the property that

the corresponding low energy supergravity theory contains the fundamental string as a supersymmetric soliton. N=1 supergravity in eleven dimensions does not contain any such soliton solution.

3. M-theory is a theory of membranes or five-branes: The eleven dimensional supergravity theory does contain membrane and five-brane like soliton solutions[15]. Thus one might argue that M-theory should be formulated as a theory of membranes or five-branes. The difficulty with this proposal is that unlike string theory, which is tractable due to the infinite dimensional conformal symmetry on its world-sheet, the world-volume theory of membranes or five-branes have no such infinite dimensional symmetry and hence are extremely difficult to handle. However, the *Matrix theory* does in some sense regard M-theory as a theory of membranes.[4]

# 6 Discrete light-cone quantization (DLCQ) and M-theory

This part was written after all the other parts were finished. I really can't resist the temptation that DLCQ gives us the possibility of doing explicit calculation in M-theory.[4]

I shall illustrate the procedure of DLCQ in the context of a scalar field theory. Let us begin with a free scalar field theory in $d+1$ dimension. The action of the system can be expressed as

$$S = \int dx^+ dx^- d^{d-1}\overrightarrow{x}_\perp (\partial_+\phi\partial_-\phi - \frac{1}{2}\sum_{i=1}^{d-1}\partial_i\phi\partial_i\phi) \qquad (57)$$

where

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^d), \qquad \partial_\pm = \frac{\partial}{\partial x^\pm} \qquad (58)$$

and $\overrightarrow{x}_\perp \equiv (x^1, ...x^{d-1})$ denotes the transverse coordinates. We shall now canonically quantize this theory by regarding $x^+$ as time. Thus the momentum conjugate to $\phi$ is

$$\pi = \frac{\delta S}{\delta(\partial_+\phi(x))} = \partial_-\phi \qquad (59)$$

Note that the relationship between $\phi$ and $\pi$ given in Eq.(59) does not involve a 'time' derivative, reflecting the fact that the original Lagrangian is linear in $\partial_+$. As a result Eq.(59) represents a constraint. The canonical equal time commutation relations can be found by using the formalism for quantization of constrained system. However, we can simplify this problem by going to the Fourier transformed variables defined through the decomposition:

$$\phi(x^+, x^-, \overrightarrow{x}_\perp) = \int_0^\infty \frac{dk_-}{\sqrt{4\pi k_-}}[a(x^+, k_-, \overrightarrow{x}_\perp)e^{-ik_-x^-} + a^*(x^+, k_-, \overrightarrow{x}_\perp)e^{ik_-x^-} \tag{60}$$

The action (57) now takes the form:

$$S = i\int dx^+ \int_0^\infty dk_- \int d^{d-1}\overrightarrow{x}_\perp [a^*(x^+, k_-, \overrightarrow{x}_\perp)\partial_+ a(x^+, k_-, \overrightarrow{x}_\perp) + ...] \tag{61}$$

where ... denote terms without any $x^+$ derivatives. Thus if we now regard $a$ as the coordinate, its canonically conjugate momentum is given by,

$$\frac{\delta S}{\delta(\partial_+ a)} = ia^* \tag{62}$$

Upon going to the quantum theory, $a^*$ should be regarded as the Hermitian conjugate of the operator $a$. This gives the following equal time commutation rules:

$$\begin{aligned}
[a(x^+, k_-, \overrightarrow{x}_\perp), a(x^+, l_-, \overrightarrow{y}_\perp)] &= 0 = [a^\dagger(x^+, k_-, \overrightarrow{x}_\perp), a^\dagger(x^+, l_-, \overrightarrow{y}_\perp)] \\
[a(x^+, k_-, \overrightarrow{x}_\perp), a^\dagger(x^+, l_-, \overrightarrow{y}_\perp)] &= \delta(k_- - l_-)\delta(\overrightarrow{x}_\perp - \overrightarrow{y}_\perp)
\end{aligned} \tag{63}$$

From this we see that $a^\dagger$ and $a$ behave respectively as creation and annihilation operators for particles carrying momentum $k_-$ located at the transverse location $\overrightarrow{x}_\perp$. Note that the argument $k_-$ of $a$ and $a^\dagger$ extends over positive values only. This can be traced to the fact that (59) is a constraint equation, hence only half of the degrees of freedom of $\phi$ are true coordinates, the other half being momenta.

We shall now compactify the light-like direction $x^-$ on a circle of radius $L$ . This of course is not a physically meaningful system since it has closed light-like curves and hence violates causality, but we simply use it as an infrared regulator and at the end take the $L \longrightarrow \infty$ limit to recover physically meaningful answers for various processes. In this case the momentum $k_-$ is quantized as:

$$k_- = \frac{n}{L} \tag{64}$$

Eqs. (60) and (61) are now modified as:

$$\begin{aligned}
\phi(x^+, x^-, \overrightarrow{x}_\perp) &= a_0(x^+, \overrightarrow{x}_\perp) + \\
&\quad \frac{1}{\sqrt{4\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} [a_n(x^+, \overrightarrow{x}_\perp) e^{-inx^-/L} + a_n^*(x^+, \overrightarrow{x}_\perp) e^{inx^-/L}] \qquad (65)\\
S &= i \int dx^+ \int d^{d-1}\overrightarrow{x}_\perp \sum_{n=1}^{\infty} (a_n^* \partial_+ a_n + ...) \qquad (66)
\end{aligned}$$

This shows that $ia_n^*$ is the momentum conjugate to $a_n$. In quantum theory $a_n^*$ will be represented by the Hermitian conjugate $a_n^\dagger$ of the operator $a_n$. This gives the following commutation relations:

$$\begin{aligned}
[a_n(x^+, \overrightarrow{x}_\perp), a_m(x^+, \overrightarrow{y}_\perp)] &= 0 = [a_n^\dagger(x^+, \overrightarrow{x}_\perp), a_m^\dagger(x^+, \overrightarrow{y}_\perp)], \\
[a_n(x^+, \overrightarrow{x}_\perp), a_m^\dagger(x^+, \overrightarrow{y}_\perp)] &= \delta_{mn} \delta(\overrightarrow{x}_\perp - \overrightarrow{y}_\perp) \qquad (67)
\end{aligned}$$

Note that the action does not contain any term containing $x^+$ derivative of $a_0$. Thus we can interpret it as the Lagrange multiplier field and integrate it out. The Hamiltonian computed from the action (57) takes the form:

$$H = \sum_{n=1}^{\infty} \frac{L}{2n} \int d^{d-1}\overrightarrow{x}_\perp \partial_i a_n^\dagger(x^+, k_-, \overrightarrow{x}_\perp) \partial_i a_n(x^+, k_-, \overrightarrow{x}_\perp) \tag{68}$$

We can define the ground state of this system as the state annihilated by all the $a_n$'s. The $p$-particle state is created by acting with $p$ of the $a^\dagger$'s on this ground state.

Let us now consider the effect of introducing interactions in the original action (57). This will add new terms to the Hamiltonian. In particular, since the interaction terms in the Lagrangian will involve $a_0$, integrating out $a_0$ will in general produce infinite series of additional terms in $H$ even if the Lagrangian itself contains only a finite number of terms. But as long as the interactions are invariant under translation, all the terms in the Hamiltonian will conserve $k_-$. Let us now focus on a sector with $k_- = N/L$. This sector contains $N$-particle states with each particle carrying momentum $1/L$, but it may also contain states with less than $N$ particles, with some particles carrying momentum larger than $1/L$ . However, this sector does not contain states with more than $N$ particles, since all particles in this description carry strictly positive $k_-$ in units of $1/L$. This suggests that the dynamics in this sector should be describable by an $N$-body *quantum mechanical Hamiltonian* $\mathcal{H}_N^{DLCQ}$. (One should distinguish this from the Hamiltonian $H$ of the second quantized theory.) $\mathcal{H}_N^{DLCQ}$, by definition, describes the complete dynamics in the $k_- = N/L$ sector. Thus for example it should correctly reproduce the spectrum, as well as all scattering amplitudes in this sector. In particular, in this description a single particle state carrying $N$ units of momentum should appear as a bound state of $N$ particles in the spectrum of $\mathcal{H}_N^{DLCQ}$. Thus $\mathcal{H}_N^{DLCQ}$ must possess an appropriate bound state of this form. Similarly it must contain in its spectrum appropriate bound states representing $p$ particle state carrying total momentum $N/L$ for all $p < N$ , and for all possible distribution of the minus component of the total momentum $N/L$ among these $p$ particles.

Given the action of an interacting quantum field theory, it should in principle be possible to construct $\mathcal{H}_N^{DLCQ}$. Conversely, if we are given $\mathcal{H}_N^{DLCQ}$ for all $N$ , we can, in principle, completely reconstruct the spectrum and the S-matrix elements of the theory. For example, calculation of a specific $n$ particle $\longrightarrow m$ particle process will involve the following steps:

1. Calculate the scattering amplitude for a set of $n$ states carrying momenta $k_-^i = n_i/L$ going to a set of $m$ states carrying momenta $l_-^i = m_i/L$ with

$$\sum_{i=1}^{n} n_i = \sum_{i=1}^{m} m_i = N \tag{69}$$

2. Now take the limit $N \longrightarrow \infty, L \longrightarrow \infty, n_i \longrightarrow \infty$ and $m_i \longrightarrow \infty$ keeping fixed

$$N/L, \qquad k_-^i = n_i/L, \qquad l_-^i = m_i/L \tag{70}$$

In this limit we shall reproduce the physical scattering amplitude involving $(m+n)$ external legs. We are of course implicitly assuming that there is an unambiguous procedure for taking the large $N$ limit.

We shall first give a general recipe for constructing $\mathcal{H}_N^{DLCQ}$ for any (compactified) string theory or M-theory. Let $\mathcal{T}$ be such a (compactified) string or M-theory. Typically $\mathcal{T}$ has one mass parameter $M$ , which can be taken to be the Planck mass for M-theory, and $(\sqrt{\alpha\prime})^{-1}$ for string theory, and a set of dimensionless parameters, *e.g.* string coupling constant, the dimensions of the compact manifold measured in units of $M^{-1}$, *etc.* We shall label all these dimensionless parameters by $\{\overrightarrow{y}\}$.Let us denote by $\mathcal{H}_N^{DLCQ}(M, L, \{\overrightarrow{y}\})$ the DLCQ Hamiltonian for this system. $L$ as usual denotes the radius of the compact light-cone direction. We now propose the following recipe for constructing $\mathcal{H}_N^{DLCQ}$.

1. Consider the same theory $\mathcal{T}$ with the same values of the dimensionless parameters $\{\overrightarrow{y}\}$, but a different value $m$ of the mass parameter. Let us compactify this theory on a space-like circle $S^1$ of radius $R$ .We shall call this theory the auxiliary theory.

2. When $R$ is small, the Kaluza-Klein modes carrying momentum along $S^1$ are heavy, and one would expect that the dynamics of this system will be described by a non-relativistic quantum mechanical Hamiltonian. Let us focus on the sector with $N$-particles each carrying momentum $1/R$ along $S^1$, and let us denote the $N$ -body Hamiltonian describing the dynamics of this system by $\mathcal{H}_N^{KK}$ (We sub-

tract off the rest mass energy $N/R$ of these particles in defining $\mathcal{H}_N^{KK}$.)

3. $\mathcal{H}_N^{DLCQ}$ is constructed from $\mathcal{H}_N^{KK}$ by taking the limit

$$\mathcal{H}_N^{DLCQ}(M, L, \{\overrightarrow{y}\}) = \lim_{R \longrightarrow 0} \mathcal{H}_N^{KK}(m = M\sqrt{L/R}, R, \{\overrightarrow{y}\}) \qquad (71)$$

   We shall first explore some of the consequences of this recipe. Later we shall discuss how this recipe might be 'derived'.

First of all, note that since the duality transformations in $\mathcal{T}$ (before compactification on $S^1$) leave the momenta along the non-compact directions unchanged, it leaves the sector with $N$ units of momentum along $S^1$ invariant after we compactify on $S^1$. As a result $\mathcal{H}_N^{KK}$, and hence $\mathcal{H}_N^{DLCQ}$ defined in (71) is expected to possess the full set of duality symmetries of $\mathcal{T}$.

Second, note that although this recipe gives $\mathcal{H}_N^{DLCQ}$ for any theory $\mathcal{T}$ , it is particularly useful for (compactified) M-theory, since in the $R \longrightarrow 0$ limit M-theory on a circle of radius $R$ gets mapped to weakly coupled type IIA string theory. States carrying momenta along $S^1$ correspond to D0 branes in the type IIA theory. Thus the recipe given above relates $\mathcal{H}_N^{DLCQ}$ to a specific weak coupling limit of the Hamiltonian describing $N$ D0 branes in type IIA string theory, which turns out to be described by N=1 supersymmetric $U(N)$ gauge theory in ten dimensions.[4]

Finally, let us give a 'derivation' of the recipe described at the beginning of this section. First of all, we note that in the auxiliary theory $\mathcal{T}$ on $S^1$ , if we multiply all the masses by some constant $\lambda$, and simultaneously multiply all the lengths by $\lambda^{-1}$, then the Hamiltonian gets multiplied by $\lambda$ due to purely dimensional reasons. This gives the following identity:

$$\mathcal{H}_N^{KK}(m, R, \{\overrightarrow{y}\}) = \lambda^{-1}\mathcal{H}_N^{KK}(\lambda m, \lambda^{-1}R, \{\overrightarrow{y}\}) \tag{72}$$

where the first and the second arguments denote respectively the overall mass scale and the radius of $S^1$ as usual. Let us now choose:

$$r = \sqrt{RL}, \qquad m = M\sqrt{L/R}, \qquad \lambda = \sqrt{R/L} = (r/L) \tag{73}$$

Substituting this in (72) we get,

$$\mathcal{H}_N^{KK}(M\sqrt{L/R}, R, \{\overrightarrow{y}\}) = \frac{L}{r}\mathcal{H}_N^{KK}(M, r, \{\overrightarrow{y}\}) \tag{74}$$

Thus the recipe (71) can now be rewritten as

$$\mathcal{H}_N^{DLCQ}(M, L, \{\overrightarrow{y}\}) = \lim_{R\longrightarrow 0}\frac{L}{r}\mathcal{H}_N^{KK}(M, r, \{\overrightarrow{y}\}) \tag{75}$$

It is this form of the identity that we shall attempt to prove.

The basic idea behind this proof is to regard the light-like circle as an infinitely boosted space-like circle of zero radius. Let us start with theory $\mathcal{T}$ compactified on a space-like circle $S^1$ of radius $r$ . If $x$ and $t$ denote the coordinate along $S^1$ and the time coordinate respectively, then we have an identification:

$$\begin{pmatrix} x \\ t \end{pmatrix} \equiv \begin{pmatrix} x \\ t \end{pmatrix} + 2\pi \begin{pmatrix} r \\ 0 \end{pmatrix} \tag{76}$$

Let us now define new coordinates $(x', t')$ and $x'^{\pm}$ as follows:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} \equiv \begin{pmatrix} x\cosh\alpha - t\sinh\alpha \\ t\cosh\alpha - x\sinh\alpha \end{pmatrix} \tag{77}$$

$$x'^{\pm} = \frac{1}{\sqrt{2}}(t' \pm x') \tag{78}$$

In this coordinate system Eq.(76) takes the form:

$$\begin{pmatrix} x'^{+} \\ x'^{-} \end{pmatrix} \equiv \begin{pmatrix} x'^{+} \\ x'^{-} \end{pmatrix} - \sqrt{2}\pi r \begin{pmatrix} -e^{-\alpha} \\ e^{\alpha} \end{pmatrix} \tag{79}$$

Now consider the limit $r \longrightarrow 0, \alpha \longrightarrow \infty$ keeping fixed

$$L \equiv \frac{r}{\sqrt{2}} e^{\alpha} \tag{80}$$

In this limit Eq.(79) reduces to

$$\begin{pmatrix} x'^{+} \\ x'^{-} \end{pmatrix} \equiv \begin{pmatrix} x'^{+} \\ x'^{-} \end{pmatrix} - 2\pi \begin{pmatrix} 0 \\ L \end{pmatrix} \tag{81}$$

This is equivalent to compactifying $x'^{-}$ on the circle of radius $L$.

Under this map, a system carrying momentum $N/r$ along $S^1$ gets mapped to a system carrying total momentum $k'^{-} = N/L$ along the $x'^{-}$ direction. Thus it is not surprising that there is a relation between the Hamiltonian describing the two systems. To find the precise relation between these two Hamiltonians, we need to study the relation between the usual time coordinate $t$ of the original theory and the light-cone time $x'^{+}$ of the boosted theory. From Eqs.(77), (78) it follows that:

$$\frac{\partial}{\partial x'^{+}} = \frac{1}{\sqrt{2}} e^{\alpha} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \tag{82}$$

Since the quantum operators which generate $i(\partial/\partial x'^{+}), i(\partial/\partial t)$ and $i(\partial/\partial x)$ are $\mathcal{H}_N^{DLCQ}$, $\mathcal{H}_N^{KK} + M_N$ and $-N/r$ respectively, with $M_N$ being the mass of the $N$ Kaluza-Klein modes in $\mathcal{T}$ on $S^1$, we see from Eq.(80), (82) that in the $r \longrightarrow 0$ limit with $L$ fixed,

$$\mathcal{H}_N^{DLCQ}(M, L, \{\overrightarrow{y}\}) = \frac{L}{r}(\mathcal{H}_N^{KK}(M, r, \{\overrightarrow{y}\}) + M_N - N/r) = \frac{L}{r}\mathcal{H}_N^{KK}(M, r, \{\overrightarrow{y}\}) \tag{83}$$

since $M_N = N/r$. This reproduces (75).

If we recall $\mathcal{H}_N^{DLCQ}$ is supposed to describe the theory $\mathcal{T}$ , whereas $\mathcal{H}_N^{KK}$ describes theory $\mathcal{T}$ compactified on a small circle, then by the above argument, quite generally we can reconstruct a theory by knowing its behavior when compactified on a small circle. This seems counterintuitive, so let us examine the steps leading to this conclusion. They may be summarized as follows:

1. We start with a small circle.
2. We convert this to an almost light-like circle of finite radius via a large boost.
3. We then take the limit where the radius of this light-like circle goes to infinity.

As we can see, the key point in this proof is the assumption that a light-like circle can be considered as a space-like circle of zero radius in the limit of infinite boost. Of course, this may be taken as a definition of the light-like circle. However, we are interested in a definition in which the radius of the light-like circle acts as an infrared regulator in the uncompactified theory, so that in the end by taking $L \longrightarrow \infty$ limit we recover the amplitudes in the uncompactified theory. Clearly there is a possibility that these two definitions do not match[4]. Indeed there are explicit computations which show that these two definitions do not always match for finite $N$. Note that the 'proof' given above did not involve taking the $N \longrightarrow \infty$ limit.) It has been suggested that this problem might go away in the $N \longrightarrow \infty$ limit. This of course does not mean that Matrix theory is wrong, it is just that we do not know for sure if it is right, and even if it is right, we do not quite know why it is right. Perhaps the arguments of Ref.[17] together with supersymmetry non-renormalization theorems and properties of the large $N$ limit can be combined to constitute such a proof.

# 7 Comments on references

Sen's review[4] is pedagogical and pretty comprehensive. It analyzes the nitty-gritty very carefully. Although It's sometimes short of detailed derivations, it's fun to think of the exact equations while reading the description in the paper. Schwarz's review[6] and his corrected paper[7] are full of the insight from a real master. He also discusses type IIB in great detail. Ref[9][8] are famous references for $E_8 \times E_8$ and SO(32).

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