The Geometry of Supersymmetric $D = 2$ Nonlinear Sigma Models

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1 Introduction

This paper will consider supersymmetric nonlinear sigma models in two space-time dimensions. First, the rich variety of possible supersymmetries in two dimensions will be discussed. Next, a slight detour into some complex geometry will be taken to gather some useful definitions and results. Finally, it will be shown that requiring the sigma model to have extended supersymmetry imposes various geometric constraints on the target manifold of the model.

Nonlinear sigma models are the quantum field theories of harmonic maps from space-time into a Riemannian manifold M. In $d = 2$, a scalar field is dimensionless so a Lagrangian of the form $\mathcal{L} = \int d^2x g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j$ is renormalizable. As any function, $g_{ij}(\phi)$, leads to a renormalizable theory, this class of field theories contains an infinite number of marginal parameters. To restrict the possible parameters, symmetries are imposed. This is generally accomplished by restricting the fields to take values in a Riemannian manifold. Classic examples are S^n and $\mathbb{C}P^n$. Supersymmetric nonlinear sigma models enlarge the set of symmetries of the theory to include supersymmetries. This can be done by writing the above Lagrangian in terms of chiral superfields, for instance. The Lagrangian for the scalar components of the superfields will then be the same as above.

One reason to study these models is their application to string theory. One can view the $d = 2$ space-time as a string world-sheet. Extended world-sheet supersymmetries have had two different uses in string theory. The heterotic string has only a gauged $(1, 1)$ or $(1, 0)$ world-sheet supersymmetry, but the corresponding $(1, 1)$ or $(1, 0)$ sigma model on a suitable background can have extra rigid world-sheet supersymmetries. This fact has played a central role in the study of compactifications of the theory. Sigma models have also played a central role in the study of $N = 2$ strings, or more generally, strings with $(2, 0)$, $(2, 1)$, or $(2, 2)$ world-sheet supersymmetry. For instance, the heterotic sigma models which describe the target spaces of $(2, 1)$ strings require a particular geometry for the target space. It is Hermitean with torsion, and the field equations imply that the curvature with torsion is self-dual in four dimensions, or satisfies generalised self-duality equations in higher dimensions. For a more thorough discussion of the applications of sigma models to string theory, see [1] and references therein. The main point is that the extended supersymmetries impose restrictions on the geometry of the target manifold. This, in turn, allows one to make stronger statements about the quantum theory, with regards to vacuum structure and ultra-violet behavior in particular.

2 The Supersymmetry Algebra of Type (p,q) in $d = 2$

Before constructing supersymmetric sigma-models, one must know the proper supersymmetry algebra in 2 dimensions. Let's carry out the discussion in generality, for the moment. For a general d dimensional space-time with t time-like directions and s space-like directions, $d = t + s$, the supercharges live in the spinor representation of $SO(t, s)$.

2.1 A Brief Discussion of Spinors

Spinors of $SO(t,s)$ have $2^{\lfloor(\frac{d}{2})\rfloor}$ complex components and transform under Lorentz via

$$\delta_L \psi = -\frac{1}{4}\lambda^{ab}[\gamma_a, \gamma_b]\psi \tag{1}$$

where the $\gamma^a, a = 1 \dots d$, satisfy the clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \tag{2}$$

However, the number of independent components of the spinors can be reduced by imposing Weyl and/or Majorana conditions. These conditions must be consistent with the Lorentz transformation law.

Weyl spinors are those having a definite chirality in even dimensions. The chirality of spinors is defined as an eigenvalue of the generalized γ^5 matrix

$$\gamma^5 = (-1)^{\frac{1}{4}(s-t)}\gamma^1\gamma^2\cdots\gamma^d. \tag{3}$$

Weyl spinors with positive (negative) chirality are defined by $\gamma^5\psi = +(-)\psi$. In odd dimensions γ^5 is proportional to the identity and one cannot define Weyl spinors [2].

Majorana spinors are those satisfying a certain kind of reality condition: $\psi^c = \psi$, where c represents charge conjugation. In even dimensions, this is defined by the following procedure. The matrices $\pm(\gamma^a)^*$ also satisfy (2), so there exists matrices $B_\pm$ which relate $\pm(\gamma^a)^*$ to γ^a by similarity transformations:

$$(\psi^a)^* = B_+\gamma^a B_+^{-1} \qquad -(\psi^a)^* = B_-\gamma^a B_-^{-1}. \tag{4}$$

The charge conjugation is defined by using one of these matrices:

$$\gamma^c = B_+^{-1}\psi^* \qquad \text{or} \;\; \gamma^c = B_-^{-1}\psi^*. \tag{5}$$

The operation is called charge conjugation because when a spinor satisfies the Dirac equation, its conjugate satisfies the Dirac equation with the opposite charge. In the case of B_- conjugation the mass term does not flip sign, whereas for B_+ is does. The matrices $B_\pm$ can be related to the usual charge conjugation matrix C by introducing the Dirac conjugate:

$$\begin{aligned}&\bar{\psi} = \psi^\dagger A, \qquad A = \gamma^1\gamma^2\cdots\gamma^t\\ &\psi^c = C_+\bar{\psi}^T \qquad \textit{or} \qquad \psi^c = C_-\bar{\psi}^T \qquad \text{with } C_\pm = B_\pm^{-1}(A^{-1})^T.\end{aligned} \tag{6}$$

For the Majorana condition to be consistent, it must be that $(\psi^c)^c = \psi$, which is equivalent to $B_+^*B_+ = 1$ or $B_-^*B_- = 1$. It can be shown [2] that the $B_\pm$ satisfy

$$B_\pm^*B_\pm = \epsilon_\pm(t,s)\mathbb{1} \qquad \text{with } \;\epsilon_\pm(t,s) = \sqrt{2}^{(s-t-1)mod(2)}\cos[\frac{\pi}{4}(s-t\pm 1)]. \tag{7}$$

Hence, Majorana spinors can only be defined when $\epsilon_+(t,s) = 1$ or $\epsilon_-(t,s) = 1$. Sometimes, spinors satisfying the reality condition with charge conjugation defined by using the matrix B_+ are called *pseudo-Majorana spinors*, while those using B_- are called Majorana spinors. Observe that $\epsilon_\pm(t,s)$ is periodic in $(s-t)$ mod 8. Its values are recorded in Table 1.

Finally, it may be possible to impose the (pseudo)-Majorana and Weyl conditions simultaneously. This will be the case when ψ and ψ^c have the same chirality. It can be shown [2] that this occurs exactly when $s - t = 0mod(8)$. For space-times with Minkowski signature, $t = 1, s = d-1$, this requirement reads $d = 2mod(8)$.

Let us now specialize to the case $d = 2$. For a Minkowski space-time $s - t = 0mod(8)$, and one may impose both the Majorana and Weyl conditions or both the pseudo-Majorana and Weyl conditions. This

s-t	1	2	3	4	5	6	7	8
ϵ_+	0	-1	-1	-1	0	1	1	1
ϵ_-	1	1	0	-1	-1	-1	0	1

Table 1: Values of $\epsilon_\pm$ as a function of $s-t$. $+1$ corresponds to space-times in which the pseudo-Majorana or Majorana condition may be imposed. Note for example, in $d=4$ Minkowski space-time, $s-t=2$ and hence it is possible to define Majorana spinors using C_- for the charge conjugation.

allows one to reduce the general spinor with two complex components to a spinor with one real component. If the space is Euclidian, then one may impose the Weyl condition or the Majorana condition, but not both. Therefore the spinor must contain two real components.

2.2 The (Twisted) Supersymmetry Algebra

The type (p,q) supersymmetry algebra in two dimensions can now be given. Central charges will not be considered, though it is easy enough to do so [2]. First, we have the usual Poincare algebra of translations, rotations, and boosts. The supercharges commute with translations and are spinors under Lorentz transformations:

$$[P_a, Q^i] = 0, \qquad [M_{ab}, Q^i] = \frac{1}{2}\gamma_{ab}Q^i. \tag{8}$$

These relations hold in arbitrary dimension with arbitrary signature. In $2mod(8)$ dimensions with Minkowski signature, the supercharges are Majorana-Weyl spinors with positive chirality $Q^i_+ (i=1,2,\cdots,p)$ and Majorana-Weyl spinors with negative chirality $Q^i_- (i=1,2,\cdots,q)$. The anti-commutation relations are

$$\begin{aligned}
\{Q^i_+, Q^{jT}_+\} &= \frac{1}{2}(1+\gamma_5)\gamma^a C_- P_a \delta^{ij}, \\
\{Q^i_-, Q^{jT}_-\} &= \frac{1}{2}(1-\gamma_5)\gamma^a C_- P_a \delta^{ij}, \\
\{Q^i_+, Q^{jT}_-\} &= 0.
\end{aligned} \tag{9}$$

In two dimensions, where the supercharges have one real component, taking $\gamma^1 = \sigma^1, \gamma^2 = \imath\sigma^2 \Rightarrow \gamma^5 = \sigma 3$, and $C_- = \mathbb{i}\sigma^2$, it is easy to show that the above reduces to

$$\begin{aligned}
\{Q^i_+, Q^j_+\} &= 2\delta^{ij}P_+, \\
\{Q^{i'}_-, Q^{j'}_-\} &= 2\delta^{i'j'}P_-, \\
\{Q^i_+, Q^{j'}_-\} &= 0,
\end{aligned} \tag{10}$$

where $i,j = 1,\cdots,p$; $i',j' = 1,\cdots,q$, and $P_\pm = (P_1 \pm P_2)$ are the momenta associated with light-cone coordinates. This has the simple interpretation that in $d=2$ it is possible to have conserved charges that are carried only by massless left-movers or only by massless right-movers because there is a Lorentz-invariant notion of whether a massless particle is moving to the left or to the right [3]. For the case of a $d=2$ space with Euclidian signature, the spinors can be taken to have one complex component. The algebra can still be put into the form (10), where the $P_\pm$ are now complex: $P_\pm = (P_1 \pm \mathbb{i}P_2)$ [1].

However, this discussion is not yet quite complete; there is one more *twist*. In $d=2$, it is possible to modify the supersymmetry algebra (10), by allowing some of the Q_+ and/or Q_- to square to $-P_+$ (or $-P_-$). Again, this is essentially made possible by the fact that the supercharges corresponding to left-moving and right-moving massless particles commute with one another. Hence, (10) should be generalized to its final form:

$$\{Q^i_+, Q^j_+\} = 2\eta^{ij} P_+,$$
$$\{Q^{i'}_-, Q^{j'}_-\} = 2\eta^{i'j'} P_-,$$
$$\{Q^i_+, Q^{j'}_-\} = 0, \tag{11}$$

where $\eta^{ij} = diag(\mathbb{1}_{uu}, -\mathbb{1}_{vv})$ with $u + v = p$ and similar remarks apply for $\eta i'j'$ [1]. Note that if $u = 0$ or $v = 0$ in the positive chirality part, say, of the algebra in (11) then there is no difference between the positive chirality parts of (11), (10). When u and v are different from zero, this algebra is referred to as a *twisted type* (p, q) *superalgebra*.

Finally, the algebra (11) can be cast in a coordinate representation in (p, q) superspace. The supersymmetry generators

$$Q^i_+ = \frac{\partial}{\partial \theta^+_i} - \mathbb{i}\eta^{ij}\theta^+_j \partial_+, \qquad Q^{i'}_- = \frac{\partial}{\partial \theta^-_{i'}} - \mathbb{i}\eta^{i'j'}\theta^-_{j'}\partial_- \tag{12}$$

satisfy the superalgebra. The corresponding supercovariant derivatives are

$$D^i_+ = \frac{\partial}{\partial \theta^+_i} + \mathbb{i}\eta^{ij}\theta^+_j \partial_+, \qquad D^{i'}_- = \frac{\partial}{\partial \theta^-_{i'}} + \mathbb{i}\eta^{i'j'}\theta^-_{j'}\partial_-. \tag{13}$$

3 Some Definitions and Results from Complex Geometry

As will be shown, various levels of supersymmetry will impose different constraints on the structure of the target space manifold for the non-linear sigma model. It will be worthwhile to summarize the necessary definitions and results needed to understand these constraints. The key concepts are the holonomy group associated with a connection, complex structures, and almost product (or *real*) structures. In the process Kahler, hyper-Kahler, pseudo-Kahler, and pseudo-hyper-Kahler manifolds will be defined. Most of the discussion will follow [5], [6], and chapter 15 of [4].

3.1 Holonomy

Let (M^n, g) be an n-dimensional Reimannian manifold, where g has signature (t, s), and let ∇ be a connection (covariant derivative), possibly with torsion, on the tangent bundle. Note that ∇ is not necessarily the Levi-Chivita connection, but can always be related to it by a torsion H. One can define parallel transport of tangent vectors along curves γ, with respect to ∇. Suppose that γ is a closed curved passing through $p \in M$. Then parallel transport defines a map, $\rho_\gamma : TM_p \to TM_p$, from tangent vectors at p to tangent vectors at p, given by taking a vector and parallelly transporting it around the closed curve γ. The map is linear because parallel transport is defined through a linear differential operator. Furthermore, the map preserves lengths, as parallel transport preserves lengths. Finally, if M is orientable, the map will also preserve orientation. Hence, $\rho_\gamma \in SO(t, s)$. Thus, parallel transport associates to each closed curve a map in $SO(t, s)$. Now consider the set of all such maps $\{\rho_\gamma : \text{contractible closed curves } \gamma \text{ through } p\}$. It is easily seen that this is a group under composition of maps. This is because, given a contractible closed curve, one can always traverse it in the other direction (*the inverse*), and given two such closed curves γ_1 and γ_2, traversing γ_1 first and then γ_2 second yields a new contractible closed curve. If M is connected, then this group is independent of the point p. The group is known as the local holonomy group $\mathcal{H}_\nabla$ associated with the connection. It is *local* because we are considering *contractible* closed curves. Finally, it has just been argued that $\mathcal{H}_\nabla \subset SO(t, s)$.

One can also define the parallel transport of tensors around closed curves. Tensors transform under the holonomy group as the appropriate tensor product of vectors. (Given a metric, we have a correspondence between vectors and their duals, so we may consider tensors of arbitrary type). Recall that a *covariantly* *constant* vector field is one that always returns to its initial value upon parallel transport around a contractible closed loop. It is easy enough to extend this definition to covariantly constant tensors. It can be shown that a covariantly constant tensor of type $(1,1)$, f^i_j commutes with all the elements of $\mathcal{H}_\nabla$:

$$\rho^i_k(\gamma) f^k_j - f^i_k \rho^k_j(\gamma) = 0 \qquad \forall \rho(\gamma) \in \mathcal{H}_\nabla. \tag{14}$$

Note that the existence of covariantly constant tensor fields implies that the holonomy group can not be all of $SO(t,s)$.

The action of the holonomy group is said to be reducible (irreducible) according to whether there is (there is not) a non-trivial invariant subspace. In the reducible case, one can show that local coordinates (ϕ^{i_1}, ϕ^{i_2}) can be chosen such the metric has the form

$$ds^2 = g_{i_1 j_1}(\phi^{k_1}) d\phi^{i_1} d\phi^{j_1} + g_{i_2 j_2}(\phi^{k_2}) d\phi^{i_2} d\phi^{j_2}. \tag{15}$$

Physically, this would correspond to a sigma model with two sets of fields which do not interact locally. Therefore, it is natural to restrict oneself to holonomy groups which are irreducible.

3.2 Complex Structures

Next, the notion of a complex structure is discussed. Consider a type $(1,1)$ tensor J at some point $p \in M$ which squares to *minus* the identity: $J^i_j J^j_k = -\delta^i_k$. The canonical example in 2 dimensions is

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{16}$$

It is clear that such a matrix is only possible in even dimensions, $n = 2m$. Furthermore, at a point $p \in M$ one can always choose local coordinates that put J in the canonical form:

$$J = \begin{pmatrix} 0 & \mathbb{1}_{mm} \\ -\mathbb{1}_{mm} & 0 \end{pmatrix}. \tag{17}$$

J, of course, can not be diagonalized over the reals, but it can be over the complexes. That is, we can change coordinates from the $x^1, \cdots, x^{2m}$ that put J in the form (17) to new coordinates $z^1, \cdots, z^m$, $\bar{z}^{\bar{1}}, \cdots, \bar{z}^{\bar{m}}$, $z^j = x^j + \mathbb{i} x^{m+j}$ which puts J in the form

$$J = \begin{pmatrix} \mathbb{i}\mathbb{1}_{mm} & 0 \\ 0 & -\mathbb{i}\mathbb{1}_{mm} \end{pmatrix}. \tag{18}$$

A tensor field, which at any *point* $p \in M$, can be put in the form (18) is called an *almost complex structure*.

Question: what is the obstruction to putting an almost complex structure J in the canonical form (18) around a *neighborhood* of any point p? One is confronted with a very similar question in general relativity. What is the obstruction to putting the metric g_{ij} is canonical form δ_{ij} in an neighborhood of any point p? (It can always be done at a single point). The answer, of course, is the curvature R_{ijkl}. It can be done if and only if the curvature vanishes. There is a similar answer in the case of an almost complex structure J. This leads one to define the *Nijenhuis tensor* of J:

$$N^k_{ij} = J^l_i(\partial_l J^k_j - \partial_j J^k_l) - J^l_j(\partial_l J^k_i - \partial_i J^k_l). \tag{19}$$

There is a difficult theorem by Newlander and Nirenberg which says that J can be put into canonical form in a neighborhood of a point $p \iff N = 0$. In this case, J is said to be *integrable* and is called a *complex structure*. What does it mean for a manifold to possess a complex structure? Well, in any local patch, there must exist the complex coordinates $(z^j, \bar{z}^{\bar{j}})$ which put J in canonical form. Furthermore, if J is to remain in canonical form on the overlap of two patches with coordinates $(z^j, \bar{z}^{\bar{j}}), (\zeta^j, \bar{\zeta}^{\bar{j}})$, it is not hard to see that this implies the coordinate transformations must be holomorphic: ζ is a function of the z and not the $\bar{z}$. But this is the definition of a *complex manifold*. Hence, a complex manifold is an (even-dimensional) Riemannian manifold with a complex structure.

Now, suppose J^i_j is a complex structure on M, and consider the type $(0,2)$ tensor $k_{ij} \equiv J_{ij} = g_{ik}J^k_j$. We say that g is *hermitean* if $k_{ij} = -k_{ji}$. (Sometimes J is also referred to as hermitean if this is the case). Using the definition of J in local coordinates, it is easy to see that a hermitean metric must be of the form $g_{ij} = g_{\bar{i}\bar{j}} = 0$ and $g_{i\bar{j}} = g_{\bar{j}i}$. If g is hermitean, then the k_{ij} are the components of a two-form called the *Kahler form.* If, in addition, the Kahler form is closed, $dk = 0$, then the manifold is a *Kahler manifold.* From the condition on k, it follows that $g_{i\bar{j}}$ must satisfy $\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}}$ and $\partial_{\bar{k}} g_{i\bar{j}} = \partial_{\bar{i}} g_{k\bar{j}}$. In particular, there exists a function K, called the Kahler potential, such that $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(\phi, \bar{\phi})$.

It is useful to examine the complex structure and Kahler condition from the point of view of holonomy. The existence of a complex structure J, ie. the vanishing of its Nijenuis tensor (19), does *not* imply its covariant constancy, or its invariance under the holonomy group. There are two reasons. First of all, $\mathcal{H}_\nabla$ is defined with respect to a connection ∇ which may have torsion. The torsion will, in general, act in a nontrivial way on the complex structure J. However, even if one is in the torsion free case, such that ∇ is the Levi-Chivita connection, J does not have to be invariant under the holonomy group. Though J is trivial in any coordinate patch, it may not be parallelly propagated back onto itself if the closed curve traverses several coordinate patches. However, if the metric is hermitean and k_{ij} is closed, then J will be invariant under the holonomy group; ie. if the manifold is Kahler, then J is invariant. Conversely, if one is in the torsion free case, and J is found to be invariant under the holonomy group, the M is necessarily Kahler.

Now suppose that M is 2m-dimensional, with Euclidian signature, and has a covariantly constant, with respect an arbitrary ∇, complex structure. J can be used to split any vector into m holomorphic and m anti-holomorphic indices. This corresponds to splitting the fundamental vector of $SO(2m)$ as $m \bigoplus \bar{m}$ of $U(m)$. Then the invariance of J under the holonomy group implies that the group actually satisfies $\mathcal{H}_\nabla \subset U(m) \subset SO(2m)$. In the general case of metric with signature (t, s), the existence of J implies $t = 2m_1, s = 2m_2$, and its invariance under holonomy implies $\mathcal{H}_\nabla \subset U(m_1, m_2) \subset SO(2m_1, 2m_2)$.

3.3 Almost Product, or Real Structures

Suppose there is a type (1,1) tensor, S, at some $p \in M$, which is not the identity, but squares to the identity. Then there are local coordinates around p which put S into the canonical form:

$$S = \begin{pmatrix} \mathbb{1}_{uu} & 0 \\ 0 & -\mathbb{1}_{vv} \end{pmatrix}, \tag{20}$$

with $u + v = n = dim(M)$. Now suppose a tensor field S on M can be put into this form at any point $p \in M$. Such a tensor will be referred to as an *almost real structure*. Again, the logical question is the following: under what conditions can S be put in the canonical form (20) in a neighborhood about any point $p \in M$? The answer is the same as before. The associated Nijenhuis tensor of S must vanish. If this is the case, S is integrable and called a *real structure* or, somewhat confusingly, an *almost product structure*.

Again, this paper will be interested in *hermitean real structures*, which satisfy $S_{ij} = -S_{ji}, S_{ij} = g_{ik}S^k_j$. However, this puts a strong restriction on the possible signatures of the metric g_{ij}. It can be shown that this forces the signature to be $(m, m), 2m = n = dim(M)$. It also implies that $u = v$ in (20). This geometry is sometimes referred to as *Kleinian* [1].

If the real structure is covariantly constant with respect to the connection ∇, then the holonomy group is reduced to $\mathcal{H}_\nabla \subset GL(m, \mathcal{R}) \subset SO(m, m)$. Finally, if there is no torsion, so that the connection is the Levi-Chivita connection, then a manifold with this local holonomy group is referred to as *pseudo-Kahler.*

3.4 Multiple Complex and/or Real Structures

One might naturally ask, can I have two distinct complex structures, two distinct real structures? Can I have arbitrary amounts of distinct complex and real structures? The answer to the last question is no, and the precise possibilities can be stated. If the holonomy group of M is irreducible, then one can apply Schur's Lemma for real representations of the holonomy group. This implies (see [5] and references therein) that the matrices which commute with all elements of $\mathcal{H}_\nabla$ form an associative division algebra over the reals. The only possibilities are the reals, the complexes, and the quaternions. Recall that the quaternions are $(1, i, j, k)$ satisfying:

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, \\ &ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j. \end{aligned} \tag{21}$$

In particular, i,j,and k satisfy a clifford algebra $\{I^a, I^b\} = -2\delta^{ab}$.

Actually, there is a small loophole in this argument, and there are two more associative algebras over the reals, which are not division algebras, that are possible [1]. This is the algebra $\mathbb{E}$, of "double numbers" with generators $(1, e)$ with $e^2 = 1$, and the algebra of "pseudo-quaternions" $(\tilde{i}, \tilde{j}, \tilde{k})$:

$$\begin{aligned} &\tilde{i}^2 = -1, \qquad \tilde{j}^2 = \tilde{k}^2 = 1, \\ &\tilde{j}\tilde{k} = -\tilde{k}\tilde{k} = -\tilde{i} \qquad \tilde{k}\tilde{i} = -\tilde{i}\tilde{k} = \tilde{j} \qquad \tilde{i}\tilde{j} = -\tilde{j}\tilde{i} = \tilde{k}. \end{aligned} \tag{22}$$

Here, $(\tilde{i}, \tilde{j}, \tilde{k})$ satisfy the clifford algebra $\{I^a, I^b\} = \eta^{ab}$, $\eta = diag(1, -1, -1)$.

The case where the algebra is the reals corresponds simply to the identiy δ^i_j being the only matrix that commutes with all elements of the holonomy group. Clearly, the complex algebra corresponds to the identity and one complex structure. $\mathbb{E}$ corresponds to the identity and one real structure. The quaternions correspond to the identity and three complex structures, where the complex structures satisfy the clifford algebra $\{I^a, I^b\} = -2\delta^{ab}$. Finally, the pseudo-quaternions correspond to two real structures and one complex structure, satisfying the clifford algebra $\{I^a, I^b\} = \eta^{ab}, \eta = diag(1, -1, -1)$. These are *all* of the possibilities for covariantly constant complex and or real structures on M.

If the metric has signature (t, s), the quaternionic structure is possible only if t, s are both divisible by 4, $t = 4m_1, s = 4m_2$. The holonomy group satisfies $\mathcal{H}_\nabla \subset USp(2m_1, 2m_2) \subset SO(4m_1, 4m_2)$, where $USp(2m_1, 2m_2)$ is the subgroup of $U(2m_1, 2m_2)$ preserving the symplectic structure. In order for the pseudo-quaternionic structure to be possible, the signature must be (m, m) with m even, $m = 2k$. Then the holonomy group satisfies $\mathcal{H}_\nabla \subset Sp(2k, \mathbb{R}) \subset SO(2k, 2k)$ [1]. Finally, in the case where the torsion vanishes, these manifolds are called *hyper-Kahler* and *pseudo-hyper-Kahler* respectively.

4 Type (p, q) (Twisted) Sigma Models: Some General Results

There are essentially two ways of finding sigma models that possess (p, q) supersymmetry. The first approach is a Noether procedure where one writes down a manifestly $(1, 1)$ supersymmetric action and finds conditions under which it has added supersymmetry. (The $(1, 0)$ case and its Noether extensions will be dealt with separately below). Alternatively, one can introduce extended (p, q) superspace. This section describes the Noether procedure in some generality, before specific cases are considered. The extended superspace formalism will be introduced as needed.

Consider the $(1, 1)$ sypersymmetric sigma model with superspace action

$$\mathcal{S}_{(1,1)} = \int d^2x d\theta^+ d\theta^- [g_{ij}(\Phi) + b_{ij}(\Phi)] D_+ \Phi^i D_- \Phi^j, \tag{23}$$

where the Φ^i are chiral superfields which can be viewed as coordinates on a d-dimensional manifold M with metric g_{ij}. b_{ij} is an antisymmetric tensor which is included because the kinetic piece $D_+\Phi^i D_-\Phi^j$ need not be symmetric in i, j. The action can be expanded into components. It can be shown that the bosonic term has the form of the standard nonlinear sigma model, while the b_{ij} is related to a torsion 3-form, H, of the manifold:

$$H_{ijk} = \frac{3}{2}\partial_{[i}b_{jk]}. \tag{24}$$

There are otherwise no restrictions placed on the manifold M for $(1,1)$ supersymmetry.

It will be useful to introduce the connections with torsion $\Gamma^{(\pm)i}_{jk}$, which are related to the usual Christoffel connection C^i_{jk} by

$$\Gamma^{(\pm)i}_{jk} = C^i_{jk} \pm H^i_{jk}. \tag{25}$$

Denote the corresponding covariant derivatives as $\nabla^{\pm}$. The curvature and Ricci tensors with torsion are defined in terms of $\Gamma^{(\pm)i}_{jk}$ in the usual way.

We now seek the conditions on the target space geometry under which the $(1,1)$ superspace action (23) is invariant under extra supersymmetries. If there are $p-1$ right-handed and $q-1$ left-handed extra supersymmetry transformations, then they must be of the form

$$\delta\Phi^i = \varepsilon^r T^i_{(+)rj} D_+\Phi^j + \varepsilon^{r'} T^i_{(-)r'j} D_-\Phi^j \tag{26}$$

for some tensors $T^i_{(+)rj}(\Phi), T^i_{(-)r'j}(\Phi)$ with $r = 1, \cdots, p-1$ and $r' = 1, \cdots, q-1$. Invariance of the action (23) requires that the tensors $T^i_{(+)rj}, T^i_{(-)r'j}$ satisfy

$$\begin{aligned} g_{ki}T^k_{(+)rj} + g_{kj}T^k_{(+)ri} &= 0, \\ g_{ki}T^k_{(-)r'j} + g_{kj}T^k_{(-)r'i} &= 0, \\ \nabla^+_k T^i_{(+)rj} = \nabla^-_k T^i_{(-)r'j} &= 0. \end{aligned} \tag{27}$$

Furthermore, if the supersymmetry transformations (26) are to satisfy the superalgebra (11) then the following conditions must hold. The matrices $T_{r(+)}$ and $T_{r'(-)}$ must satisfy the anti-commutation relations

$$T_{(+)r}, T_{(+)s} = -2\eta_{rs}, \qquad T_{(-)r'}, T_{(-)s'} = -2\eta_{r's'} \tag{28}$$

for some metrics η_{rs} and $\eta_{r's'}$, which, at any given point on the manifold, one can take to be diagonalized with entries ± 1. This guarantees that the commutator of two supersymmetry transformations closes into a translation. However, this point-wise condition alone is not enough to guarantee that the extended supersymmetry variations (26) can be made consistent with the supersymmetry algebra (11) everywhere. The additional conditions involve, as one might expect, the Nijenhuis tensors. The exact condition is that the generalized Nijenhuis concomitants $\mathcal{N}(T^r_+, T^s_+)$ and $\mathcal{N}(T^{r'}_-, T^{s'}_-)$ must vanish [1]. For any two $(1,1)$ tensors, T_1 and T_2, the generalized Nijenhuis concomitant is defined by

$$\mathcal{N}(T_1, T_2)^i_{jk} = T^l_{1j}\partial_l T^i_{2k} - T^l_{1k}\partial_l T^i_{2j} - T^i_{1l}\partial_j T^l_{2k} - T^i_{1l}\partial_k T^l_{2j} + (1 \to 2). \tag{29}$$

Note that $\frac{1}{4}\mathcal{N}(T,T) \equiv \mathcal{N}(T)$ is the usual Nijenhuis tensor of T.

If all of the above conditions are satisfied, then the supersymmetry transformations (26) together with the manifest $(1,1)$ supersymmetries satisfy the algebra (11) with

$$\eta^{ij} = diag(1, \eta^{rs}), \qquad \eta^{i'j'} = diag(1, \eta^{r's'}). \tag{30}$$

Target Signature	Holonomy of ∇^+	Geometry (*with torsion*)	Supersymmetry
(d_1, d_2)	$O(d_1, d_2)$	no restriction	$(1,1),(1,0),$
$(2n_1, 2n_2)$	$U(n_1, n_2)$	1 complex Hermitean structure	$(1,1),(1,0),$
$(4m_1, 4m_2)$	$USp(2m_1, 2m_2)$	3 complex Hermitean str. (quaternionic)	$(4,1),(4,0),$
$(2n, 2n)$	$GL(n, \mathbb{R})$	1 real Hermitean structure	$twisted(2,1),(2,0),$
$(4m, 4m)$	$Sp(2m, \mathbb{R})$	2 real and 1 cmplx Hrmtn. str. (pseudo-qtrnc)	$twisted(4,1),(4,0),$

Table 2: The possible $(p,1)$ and $(p,0)$ twisted and untwisted supersymmetries, where the target manifold has torsion. The first column contains the allowable signatures of the target manifold. The second column is the (maximal) local holonomy group for each case, where the holonomy is with respect to the connection with torsion. The third column lists the complex and/or real structures on the manifold. In the case of multiple structures, the algebra given refers to that of the structures plus the identity. The last column contains the applicable supersymmetries. Note that it is not possible to have the $(p,1), p>1$ supersymmetries if the manifold does not have torsion.

Now, the equations (28) imply that the $T_{(+)r}$ and $T_{(-)r'}$ are almost complex structures, or almost real structures. The vanishing of the Nijenhuis tensors imply that $T_{(+)r}$ and $T_{(-)r'}$ are complex structures and/or real structures of the manifold, depending on the corresponding sign of η_{rs} and $\eta_{r's'}$. Further, (27) imply that the $T_{(+)r}$ and $T_{(-)r'}$ are hermitean and invariants under the local holonomy group $\mathcal{H}_\nabla$ of M. Now here is where the complex geometry really pays off. The point of this exercise is that one can not have arbitrary numbers of matrices $T_{(+)r}$ and $T_{(-)r'}$. For instance, consider the positive chirality supersymmetries of (11), corresponding to the $T_{(+)r}$, and suppose for the moment that the T_- vanish. The possibilities are (1) no $T_+ \Rightarrow (1,1)$ Susy; (2) one complex structure $J_+ \Rightarrow (2,1)$ untwisted Susy; (3) one real structure $S_+ \Rightarrow (2,1)$ twisted Susy; (4) three complex structures $J_{(+)r} \Rightarrow (4,1)$ untwisted Susy; (5) one complex structure J_+ and two real structures $S_{(+)r} \Rightarrow (4,1)$ twisted Susy. The corresponding holonomy groups $\mathcal{H}_{\nabla^+}$ were discussed in the previous section. Similar remarks apply to the case $p=1, q>1$. The possibilities are again $q=2$ twisted or untwisted and $q=4$ twisted or untwisted. (There is no difference here; it is just a question of what you call left or right). These results are summarized in Table 2. This is not nearly all one can say about these theories; however, this is supposed to be a paper and not a book!

In the case where $p>1$ and $q>1$, and the torsion is non-zero, there are two separate holonomy groups $\mathcal{H}_{\nabla^\pm}$ to deal with, and the analysis is more subtle. These cases will be considered below. However, in the case of vanishing torsion, some immediate conclusions can be drawn. In this case, the connections are the same and equal to the Levi-Chivita connection $\nabla^+ = \nabla^- \equiv \nabla_{LC}$. The holonomy groups coincide, as do the complex/real structures $T_{(+)r} = T_{(-)r'} \equiv T_r$. In particular, $r=r'$, ie. $p=q$, so the number of left and right supercharges is the same. (Note, this is why the torsion free case was not considered in the preceding paragraph: if $p \neq q$ (and $p,q>0$), then the torsion must be non-zero). As the holonomy is now defined with respect to the Levi-Chivita connection, the possible geometries have names: Kahler, pseudo-Kahler, hyper-Kahler, and pseudo-hyper-Kahler. These results are summarized in Table 3. The originally studied $(2,2)$ Kahler and $(4,4)$ hyper-Kahler cases are often referred to as the $N=2$ and $N=4$ supersymmetric models in older literature.

5 Type $(p,0)$ (Twisted) Sigma Models

The type $(p,0)$, or equivalently, the type $(0,q)$ sigma models fell outside the range of previous discussions. The analysis is similar. Start with a manifestly $(1,0)$ supersymmetric action

Target Signature	Holonomy of ∇	Geometry (*without torsion*)	Supersymmetry
$(2n_1, 2n_2)$	$U(n_1, n_2)$	Kahler	$(2,2),(2,0)$
$(4m_1, 4m_2)$	$USp(2m_1, 2m_2)$	hyper-Kahler	$(4,4),(4,0)$
$(2n, 2n)$	$GL(n, \mathbb{R})$	pseudo-Kahler	$twisted(2,2),(2,0)$
$(4m, 4m)$	$Sp(2m, \mathbb{R})$	pseudo-hyper-Kahler $twstd(4,4),(4,0)$	

Table 3: The possible (p,p) and $(p,0)$ twisted and untwisted supersymmetries, where the target manifold is torsion free. Note that the torsion free condition forces, $p = q$ for supersymmetries (p,q) with $p,q > 0$.

$$\mathcal{S}_{(1,0)} = \mathbb{i} \int d^2x d\theta [g_{ij}(\Phi) + b_{ij}(\Phi)] D\Phi^i \partial_- \Phi^j, \tag{31}$$

where the + superscript on θ and D has been dropped. This is sometimes referred to as $N = 1/2$ supersymmetry. The superfield Φ has a rather short expansion:

$$\Phi(x^\mu, \theta) = \phi(x^\mu) + \theta\lambda(x^\mu), \tag{32}$$

where $\phi(x^\mu)$ is a real scalar field and $\lambda(x^\mu)$ is a left-handed Majorana-Weyl spinor for Minkowski signature space-time and a left-handed Weyl spinor for Euclidian signature. On expanding into components and doing the θ integration (31) becomes [3]

$$\mathcal{S}_{(1,0)} = \int d^2x [(g_{ij}(\phi) + b_{ij}(\phi))\partial_+\phi^i \partial_-\phi^j + \mathbb{i} g_{ij}(\phi)\lambda^i(\partial_-\lambda^j + \Gamma^j_{kl}\partial_-\phi^k\lambda^l)], \tag{33}$$

where the connection Γ differs from the Christoffel connection by the torsion, which is related to b_{ij} as before.

Extra supersymmetries must be of the form

$$\delta\phi^i = \varepsilon^r T^i_{rj}\lambda^j \qquad \delta(T^i_{rj}\lambda^j) = -\mathbb{i}\varepsilon^r\partial_+\phi^i. \tag{34}$$

In order for these supersymmetries to leave the action invariant, they must satisfy

$$g_{ki}T^k_{rj} + g_{kj}T^k_{ri} = 0, \qquad \nabla_k T^i_{rj} = 0. \tag{35}$$

In order to anti-commute with one another and satisfy the supersymmetry algebra (11) they must satisfy

$$T_r, T_s = -2\eta_{rs}, \qquad \mathcal{N}(T_r, T_s) = 0. \tag{36}$$

One is led to the same conclusions as above about the possible types of tensors T_r. In fact, as far as the geometrical structure of the target manifold is concerned, the $(p,0)$ and $(p,1)$ cases are equivalent when torsion is present here. If the torsion vanishes, then the structure of the manifold is (a) Kahler for untwisted $(2,0)$ supersymmetry, (b) pseudo-Kahler for twisted $(2,0)$ supersymmetry, (c) hyper-Kahler for $(4,0)$ untwisted supersymmetry, and (d) pseudo-hyper-Kahler for twisted $(4,0)$ supersymmetry. In this case, the $(p,0)$ geometrical structure is equivalent to the (p,p) geometrical structure.

6 (2,2) Supersymmetry With Torsion

It is possible, and preferable, to treat this model in an extended superspace formulation. We consider the cases of untwisted and twisted supersymmetry seperately.

6.1 The Untwisted Case

As the target manifold, M, must be complex, introduce complex coordinates $(z^{\alpha}, \bar{z}^{\bar{\alpha}})$, $\alpha = 1, \cdots, m$, $dim(M) = 2m$. Let θ^{+}, θ^{-} be complex Grassmann variables. Then we have the supersymmetry generators and supercovariant derivatives

$$Q_{+} = \frac{\partial}{\partial\theta^{+}} - \mathbb{i}\bar{\theta}^{+}\partial_{+}, \qquad Q_{-} = \frac{\partial}{\partial\theta^{-}} - \mathbb{i}\bar{\theta}^{-}\partial_{-} \tag{37}$$

and

$$D_{+} = \frac{\partial}{\partial\theta^{+}} + \mathbb{i}\bar{\theta}^{+}\partial_{+}, \qquad D_{-} = \frac{\partial}{\partial\theta^{-}} + \mathbb{i}\bar{\theta}^{-}\partial_{-}. \tag{38}$$

Next, the superfields must be specified. We now have four Grassmann variables, so a general superfield would have as many components as the $D = 4, N = 1$ superfields of Wess and Bagger [7]. In order to get irreducible representations of the supersymmetry algebra, one must impose contraints on the superfields. One may introduce chiral superfields $U^{\alpha}, \bar{U}^{\bar{\alpha}}; \alpha = 1, \cdots, m$ satisfying

$$\bar{D}_{\pm}U^{\alpha} = 0 \qquad D_{\pm}\bar{U}^{\bar{\beta}} = 0. \tag{39}$$

Then the action is

$$S = \int d^2x d^4\theta K(\bar{U}, U). \tag{40}$$

Upon expanding into components and solving the equations of motion for the auxiliary term, one finds a result [8] which is identical in form to the well known result in $D = 4, N = 1$:

$$\mathcal{L} = g_{i\bar{j}}(\partial_{\mu}z^{i}\partial^{\mu}\bar{z}^{\bar{i}} - \frac{1}{2}\mathbb{i}\bar{\psi}^{\bar{j}}\gamma\cdot\mathcal{D}\psi^{i} + \frac{1}{2}\mathbb{i}\bar{\psi}^{\bar{j}}\gamma\cdot\overleftarrow{\mathcal{D}}\psi^{i}) + \frac{1}{4}R_{i\bar{j}k\bar{l}}(\bar{\psi}^{\bar{j}}\bar{\psi}^{\bar{l}})(\psi^{i}\psi^{k}). \tag{41}$$

Here, the metric $g_{i\bar{j}}(z,\bar{z})$ is related to $K(U,\bar{U})$ by $g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K(z,\bar{z})$. This means, of course, that the manifold is Kahler, K being the Kahler potential. The covariant derivative and curvature in (41) are that of the Kahler manifold.

But wait, what about the torsion? Clearly, from the form of (41), there is no torsion in this model. So, how does one implement torsion in the extended superspace formulation? The answer lies in the fact, that in $d = 2$ dimensions, there is another, inequivalent constraint that can be imposed on scalar superfields. This is known, unfortunately, as a *twisted chiral constraint* leading to *twisted* chiral superfields [6]. This "twisted" has *nothing* to do with the twisted supersymmetry! Twisted chiral superfields are those satisfying

$$D_{+}V^{i} = 0 \qquad \bar{D}_{-}V^{i} = 0 \qquad D_{-}\bar{V}^{\bar{j}} = 0 \qquad \bar{D}_{+}\bar{V}^{\bar{j}} = 0. \tag{42}$$

These constraints are consistent in two, but not higher, dimensions because $D_{+}, \bar{D}_{-} = 0$ only if $d \leq 2$. In theories with *only* twisted chiral superfields, D and $\bar{D}$ can not be distinguished, and V is equivalent to an ordinary chiral superfield U satisfying $\bar{D}_{\pm}U = 0$. However, in models with both chiral and twisted chiral superfields, U and V are distinct [6]. Therefore, the action (40) should be generalized to

$$S = \int d^2x d^4\theta K(\bar{U}, U, \bar{V}, V), \tag{43}$$

which is invariant under generalized Kahler gauge transformations

$$\delta K = f_1(U,V) + f_2(U,\bar{V}) + \bar{f}_1(\bar{U},\bar{V}) + \bar{f}_2(\bar{U},V). \tag{44}$$

Now, let the coordinates on the manifold (the lowest components of U, V) be labeled $z^{J} = (u^{\alpha}, \bar{u}^{\bar{\alpha}}, v^{j}, \bar{v}^{\bar{j}})$ where $\alpha = 1, \cdots, m_1$ and $i = 1, \cdots, m_2$. That is, J runs over all of $(\alpha, \bar{\alpha}, i, \bar{i})$. The manifold is complex with dimension $n = 2m_1 + 2m_2$. Then, it can be shown [1] that the bosonic part of the component sigma model action is

$$S_{bos} = \frac{1}{2}\int d^2x(g_{JK}\partial_\mu z^J\partial^\mu z^K + b_{JK}\epsilon^{\mu\nu}\partial_\mu z^J\partial_\nu z^K). \tag{45}$$

Observe the reappearance of the torsion! The metric g_{MN} and the torsion potential b_{MN} are given in terms of the generalized Kahler potential by

$$\begin{aligned} g_{\alpha\bar\beta} &= \partial_\alpha\partial_{\bar\beta}K, \qquad g_{i\bar j} = -\partial_i\partial_{\bar j}K, \\ b_{\alpha\bar j} &= \partial_\alpha\partial_{\bar j}K, \qquad b_{i\bar\beta} = \partial_i\partial_{\bar\beta}K. \end{aligned} \tag{46}$$

All other components of g_{MN} and b_{MN} not related to these by complex conjugation or symmetry vanish. In the special case that either $m_1 = 0$ or $m_2 = 0$, the torsion vanishes and the target space is Kahler.

The question to be posed now is, what's the geometry of the target space M? From the general analysis of Section 4, one knows that for $(2,2)$ untwisted supersymmetry, M must have two complex structures $T_\pm$, convariantly constant with respect to the connections $\nabla^\pm$. Furthermore, from the superalgebra (11), T_+ and T_- commute. Given two such commuting structures, the tensor

$$S^N_M = -T^K_{(+)M}T^N_{(-)K} \tag{47}$$

satisfies

$$S^K_M S^N_K = \delta^N_M \tag{48}$$

and has vanishing Nijenhuis tensor. Hence, S is a real structure. The fact that g_{MN} is hermitian with respect to both T_+ and T_- implies that

$$S^N_M g_{NP} = S^N_P g_{NM} \qquad or \qquad S_{MP} = S_{PM}. \tag{49}$$

Note that this is different than the hermitean, anti-symmetric, requirement put on S in Sections 3 and 4. The commutivity of the $T_\pm$, the hermiticity of g with respect to $T_\pm$, the existence of S, and the property (49), all come together to imply that there exists local complex coordinates (u^α, v^j), $\alpha = 1,\cdots,m_1$, $j = 1,\cdots,m_2$ such that g_{MN} can be put exactly in the form required by (46). In this form, g is block diagonal, the blocks being m_1 by m_1 and m_2 by m_2. Furthermore, in each block, g is off-diagonal and hermitean:

$$g_{MN} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & g_{ij} \end{pmatrix}, \qquad g_{\alpha\beta} = \begin{pmatrix} 0 & g_{\alpha\bar\beta} \\ g_{\bar\beta,\alpha} & 0 \end{pmatrix}, \qquad g_{ij} = \begin{pmatrix} 0 & g_{i\bar j} \\ g_{\bar j,i} & 0 \end{pmatrix}. \tag{50}$$

For obvious reasons, a manifold that admits such a metric is called a *locally hermitean product space* [6]. It is *local* because each component of g in (50) will in general depend on all of the coordinates. If it depended only on its respective coordinates everywhere on the manifold, then M would simply be a (hermitean) product space. In fact, it follows from (46) that the submanifolds projected out by $\frac{1}{2}(\delta^N_M \pm S^N_M)$ are Kahler.

6.2 The Twisted Case

Here, the case of twisted $(2,2)$ supersymmetry is considered. A restriction to the case where both positive and negative chirality supersymmetries are twisted is made. The analysis of the mixed case will be completely obvious after the following. Introduce the real Grassmann parameters $(\theta^+, \tilde\theta^+, \theta^-, \tilde\theta^-)$. The supercharges are real Majorana spinors

$$Q_+ = \frac{\partial}{\partial\theta^+} - \mathbb{i}\tilde\theta^+\partial_+, \qquad Q_- = \frac{\partial}{\partial\theta^-} - \mathbb{i}\tilde\theta^-\partial_- \tag{51}$$

Supersymmetry	Superfields	Target Geometry (*with torsion*)
$(2,2)untwisted$	chiral field U and twisted chiral field $\bar{V}$	Hermitean locally product space w/ 2 commuting complex structures
$(2,2)twisted$	chiral field U and twisted chiral field $\tilde{V}$	real locally product space w/ 2 commuting real structures

Table 4: The geometries for type $(2,2)$ untwisted and twisted supersymmetry, where the target manifold has torsion. The models were constructed with an extended superspace formulation. In order to achieve torsion on the target space, "twisted" chiral multiplets were introducted, satisfying "twisted" chiral conditions.

and the covariant derivatives are

$$D_+ = \frac{\partial}{\partial \theta^+} + \mathbb{i}\tilde{\theta}^+\partial_+, \qquad D_- = \frac{\partial}{\partial \theta^-} + \mathbb{i}\tilde{\theta}^-\partial_-. \tag{52}$$

Again, one can introduce superfields $U^\alpha, \tilde{U}^{\tilde{\alpha}}$ satisfying chiral constraints

$$\tilde{D}_\pm U^\alpha = 0 \qquad D_\pm \tilde{U}^{\tilde{\beta}} = 0 \tag{53}$$

or superfields $V^i, \tilde{V}^{\tilde{i}}$ satisfying twisted chiral constraints

$$D_+\tilde{V}^{\tilde{i}} = 0 \qquad \tilde{D}_-\tilde{V}^{\tilde{i}} = 0 \qquad D_-V^j = 0 \qquad \tilde{D}_+V^j = 0. \tag{54}$$

The action

$$S = \int d^2x d^2\theta d^2\tilde{\theta} K(\tilde{U}, U). \tag{55}$$

leads to the pseudo-Kahler sigma model. The action

$$S = \int d^2x d^2\theta d^2\tilde{\theta} K(\tilde{U}, U, \tilde{V}, V), \tag{56}$$

defines a supersymmetric non-linear sigma model with torsion on a target space of dimension $2(m_1 + m_2)$ with real coordinates $(u, \tilde{u}, v, \tilde{v})$. It is invariant under generalized pseudo-Kahler gauge transformations

$$\delta K = f_1(U, V) + f_2(U, \tilde{V}) + \tilde{f}_1(\tilde{U}, \tilde{V}) + \tilde{f}_2(\tilde{U}, V). \tag{57}$$

The bosonic part of the action is again given by (45), where the metric and torsion potential are given by

$$\begin{aligned} g_{\alpha\tilde{\beta}} &= \partial_\alpha\partial_{\tilde{\beta}}K, \qquad g_{i\tilde{j}} = -\partial_i\partial_{\tilde{j}}K, \\ b_{\alpha\tilde{j}} &= \partial_\alpha\partial_{\tilde{j}}K, \qquad b_{i\tilde{\beta}} = \partial_i\partial_{\tilde{\beta}}K. \end{aligned} \tag{58}$$

All other components not related by "real" conjugation or symmetry vanish. From the form of the metric, we infer that the geometry is that of a *locally real product space* with two *real* commuting structures $S_\pm$. This concludes the discussion of $(2,2)$ supersymmetry. The results are summarized in Table 4.

7 A Few Comments About $(4,2)$ and $(4,4)$ Supersymmetries with Torsion

First note that it is not possible to write down an extended superspace formulation of these models that has the full supersymmetry manifest. The measure would involve six or eight spinor derivatives, and has the wrong dimension to be of use. It is possible, however, to formulate the models in terms of restricted measures with four spinor derivatives. The interested reader is referred to [6].

The key point to be made here is that in these models, there will be multiple complex/real structures. The positive and negative chirality sets will commute with each other. For every pair of commuting structures, a new real structure can be defined, as in the previous section. For example, the $(4,4)$ untwisted case has two commuting quaternionic structures. The complex structures generate nine real structures. It can be shown [6] that the six complex structures and nine real structures satify an $SL(4,\mathbb{R})$ algebra, the $SO(4) \simeq SU(2) \times SU(2)$ subalgebra being generated by the six complex structures. In general there are several possibilities depending the amount of twisting in the $(4,2)$ and $(4,4)$ supersymmetry.

8 Conclusions

The general (p,q) twisted supersymmetry in $d = 2$ dimensions, (without central charges), was discussed. In the case of Minkowski signature, the supercharges may be taken as one-component, real Majorana-Weyl spinors. In the case of Euclidian signature, they may be taken as one-component complex Weyl spinors or two-component real Majorana spinors. Some results and definitions from complex geometry were given. In particular, the important notions of homonomy, complex structures, and real structures were considered. It was then shown that the amount of extended supersymmetry possible is constrained purely by geometric considerations. The possible values of p, q are $0, 1, 2$, and 4. In the case of 2 and 4, one is permitted the twisted or untwisted supersymmetry. The geometries of the allowed cases were then classified. In the classic cases of $p = q$ untwisted supersymmetry, with a torsion-free target space, sometimes referred to as $N = 1, 2$, and 4 supersymmetry, it was shown the the manifold structure is unrestricted for $N = 1$, Kahler for $N = 2$, and hyper-Kahler for $N = 4$. For general cases, results are summarized in Tables 2, 3, and 4. It is notable that any (p,q) symmetry with $p \neq q$, $p, q > 0$ necessarily has torsion. In the case of $(2,2)$ supersymmetry, it was shown how the metric and torsion can be related to a generalized Kahler or pseudo-Kahler potential.

The obvious shortcomings of this paper are in regard to the quantum structures of the theories. The geometrical structures required by the supersymmetries can, in turn, put strong constraints on the quantum structure of the model. For instance, it can be shown [1] that the general type (p,q) model will be conformally invariant at one-loop if there is a function Φ such that

$$R^{+}_{ij} - \nabla^{+}_{(i}\nabla^{+}_{j)}\Phi - H^{k}_{ij}\nabla^{+}_{k}\Phi = 0. \tag{59}$$

where R^{+}_{ij} and ∇^{+}_{k} are the Ricci tensor and covariant derivative with torsion, the torsion given by H^{k}_{ij}. In the particular case of the $(2,1)$ untwisted model, it can be shown that geometries for which $\Gamma^{+}_{i} = 0$ in some suitable choice of coordinate system will satisfy (59) provided Φ, the *dilaton field*, is chosen as

$$\Phi = -\frac{1}{2} \lg \det g_{\alpha\bar{\beta}}. \tag{60}$$

It is regrettable that this paper was not able to cover any of these results pertaining to the quantum structures of the models. It is hoped, however, that a solid foundation has been laid here, from which these issues could be pursued with confidence.

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