$N = 2$ supersymmetric Yang-Mills Theory

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In this final project, we will report our understanding of Seiberg Witten theory [1][2]. The low energy effective theory of $N = 2$ supersymmetric Yang-mill theory has a non-trivial quantum moduli space. And the structure of the moduli space can be exactly determined by the property near the singularities of the moduli space. This theory also provide a example for electric-magnetic duality. In the first four section, we discuss pure gauge theory. In the last section, we briefly discuss gauge theory with matter.

1 $N = 2$ Supersymmetric Lagrangian

We have learned the most general form of $N = 1$ supersymmetric Lagrangian as

$$\mathcal{L} = \frac{1}{8\pi}\,\mathrm{Im}\left(\tau\mathrm{Tr}\int d\theta\, W^\alpha W_\alpha\right) + \int d^2\theta d^2\bar{\theta}\,\Phi^\dagger e^{-2V}\Phi + \int d^2\theta\,\mathcal{W} + \int d^2\bar{\theta}\,\bar{\mathcal{W}}\,. \tag{1}$$

with $\tau = \theta/2\pi + 4\pi i/g^2$. τ can be regarded as a constant chiral superfield

In component form, we have

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\widetilde{F}^{a\mu\nu} - \frac{i}{g^2}\,\lambda^a\sigma^\mu D_\mu\bar{\lambda}^a + \frac{1}{2g^2}D^aD^a \\
& + (\partial_\mu A - iA^a_\mu T^a A)^\dagger(\partial^\mu A - iA^{a\mu}T^a A) - i\,\bar{\psi}\bar{\sigma}^\mu(\partial_\mu\psi - iA^a_\mu T^a\psi) \\
& - D^aA^\dagger T^a A - i\sqrt{2}\,A^\dagger T^a\lambda^a\psi + i\sqrt{2}\,\bar{\psi}T^aA\bar{\lambda}^a + F_i^\dagger F_i \\
& + \frac{\partial\mathcal{W}}{\partial A_i}F_i + \frac{\partial\bar{\mathcal{W}}}{\partial A_i^\dagger}F_i^\dagger - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial A_i\partial A_j}\,\psi_i\psi_j - \frac{1}{2}\frac{\partial^2\bar{\mathcal{W}}}{\partial A_i^\dagger\partial A_j^\dagger}\,\bar{\psi}_i\bar{\psi}_j\,.
\end{aligned} \tag{2}$$

We can put the on-shell $N = 1$ scalar multiplet (A, ψ) and vector multiplet (A_μ, λ) together to have get on-shell $N = 2$ vector multiplet $(A, \psi, \lambda, A_\mu)$. $N = 2$ supersymmetric will impose some restriction on on the $N = 1$ Lagrangian in (2). First, since A^a_μ and λ^a belong to the adjoint representation of the gauge group, A_i and ψ_i should also belong to the same representation if they are to be part of the same multiplet. Second, We must treat the fermions ψ^a and λ^a on the same footing. Thus the superpotential $\mathcal{W}$ must be zero since it couples only to ψ^a. This also requires that the kinetic energy terms for both fermions should have the same normalization. Thus we must take scaling $\Phi \to \Phi/g$. Then we get the $N=2$ supersymmetry Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{8\pi}\mathrm{Im}\,\mathrm{Tr}\left[\tau\left(\int d^2\theta\, W^\alpha W_\alpha + 2\int d^2\theta d^2\bar\theta\, \Phi^\dagger e^{-2V}\Phi\right)\right] \\
&= \frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g^2\frac{\theta}{32\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu} + (D_\mu A)^\dagger D^\mu A - \frac{1}{2}[A^\dagger\,,\,A]^2 \\
&\qquad - i\,\lambda\sigma^\mu D_\mu\bar\lambda - i\,\bar\psi\bar\sigma^\mu D_\mu\psi - i\sqrt{2}\,[\lambda,\psi]\,A^\dagger - i\sqrt{2}\,[\bar\lambda,\bar\psi]\,A\Big)\,, \qquad (3)
\end{aligned}
$$

with scalar potential

$$
V = -\frac{1}{2g^2}\mathrm{Tr}\left([A^\dagger\,,\,A]^2\right)\,. \qquad (4)
$$

This can also be obtained by $N=2$ superspace formalism. Introduce two new fermionic coordinates $\tilde\theta$ and $\bar{\tilde\theta}$, We can define $N=2$ chiral superfield Ψ satisfying the constraints $\bar D_{\dot\alpha}\Psi = 0$ and $\bar{\tilde D}_{\dot\alpha}\Psi = 0$. It can be expand as

$$
\Psi = \Psi^{(1)}(\tilde y,\theta) + \sqrt{2}\tilde\theta^\alpha\Psi^{(2)}_\alpha(\tilde y,\theta) + \tilde\theta^\alpha\tilde\theta_\alpha\Psi^{(3)}(\tilde y,\theta)\,,
$$

with $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta + i\tilde\theta$. the component $\Psi^{(1)}$ has the same form as the $N=1$ chiral superfield Φ. And we have

$$
\begin{aligned}
\Psi^{(2)}_\alpha &= W_\alpha(\tilde y,\theta) \\
\Psi^{(3)}(\tilde y,\theta) &= \Phi^\dagger(\tilde y - i\theta\sigma\bar\theta,\theta,\bar\theta)exp\left[2gV(\tilde y - i\theta\sigma\bar\theta,\theta,\bar\theta\right]\,|_{\bar\theta\bar\theta}\ .
\end{aligned}
$$

With these components, we can write (3) in terms $N=2$ chiral superfield Φ.

$$
\mathcal{L} = \frac{1}{4\pi}\mathrm{Im}\,\mathrm{Tr}\int d^2\theta d^2\tilde\theta\,\frac{1}{2}\tau\Psi^2\,. \qquad (5)
$$

We can generalize the quadratic form to an arbitrary function, then we have Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{4\pi}\mathrm{Im}\,\mathrm{Tr}\int d^2\theta d^2\tilde\theta\,\mathcal{F}(\Psi) \\
&= \frac{1}{8\pi}\mathrm{Im}\left(\int d^2\theta\,\mathcal{F}_{ab}(\Phi)W^{a\alpha}W^b_\alpha + 2\int d^2\theta d^2\bar\theta\,(\Phi^\dagger e^{2gV})^a\mathcal{F}_a(\Phi)\right)\,. \qquad (6)
\end{aligned}
$$

Here, $\mathcal{F}_a(\Phi) = \partial\mathcal{F}/\partial\Phi^a$, $\mathcal{F}_{ab}(\Phi) = \partial^2\mathcal{F}/\partial\Phi^a\partial\Phi^b$ and $\mathcal{F}$ is called the $N=2$ prepotential.

The above is pure gauge theory. We can couple it with $N=2$ matter supermutiplet, also call hypermultiplet.In $N=1$ notation, a hypermultiplet contains a chiral superfield Q and an anti-chiral superfield $\widetilde{Q}^\dagger$, both transforming under the same representatin N_c of the gauge group $SU(N_c)$ with components (q,ψ_q,F_q) and $(\tilde q,\psi_{\widetilde q},F_{\widetilde q})$ respectively. The form of the Lagrangian for N_f hypermultiplets (labelled by an index i), interacting with a $N=2$ vector multiplet, is given by

$$
\mathcal{L} = \int d\theta^4\left(Q_i^\dagger e^{-2V}Q_i + \widetilde{Q}_i e^{2V}\widetilde{Q}_i^\dagger\right) + \int d\theta^2\left(\sqrt{2}\widetilde{Q}_i\Phi Q_i + m_i\widetilde{Q}_i Q_i\right) + h.c. \qquad (7)
$$

2 Low-Energy Effective Theory

for now we only consider the pure Yang-mill case. The theory has gauge symmetry $SU(2)$, and the $U(1)$ R-symmetry acts on the superspace coordinates as $\theta \rightarrow e^{i\alpha}\theta$ and $\bar{\theta} \rightarrow e^{-i\alpha}\bar{\theta}$.Under this symmetry, chiral superfield transform as $\Phi \rightarrow e^{2i\alpha}\Phi(e^{-i\alpha}\theta)$ and $V \rightarrow V(e^{-i\alpha}\theta)$. Thus the fermion components transform like $(\psi, \lambda) \rightarrow e^{i\alpha}(\psi, \lambda)$. We can combine the two-component spinors λ and $\bar{\psi}$ into a four-component Dirac spinor ψ_D. The spinor ψ_D transforms as $e^{i\alpha\gamma_5}\psi_D$ under under $U(1)_{\mathcal{R}}$. Thus this symmetry has anomaly in quantum theory. Anomaly will break this R-symmetry to a discrete group $\mathbf{Z}_8$. This will be explained later. We also have another R-symmetry $SU(2)_R$ which rotating the two sets of superspace coordinates.

The scalar potential of this model is

$$V = \frac{1}{2g^2}\mathrm{Tr}\left([\phi^\dagger, \phi]^2\right) .$$

On classical level, the vacuum state must satisfy $V = 0$. This implies that ϕ takes values in the Cartan subalgebra of the gauge group. Thus we can get $\phi = \frac{1}{2}a\sigma_3$. Since the Weyl reflection of the gauge group will relate a to $-a$, we can use $u = \mathrm{Tr}(\phi^2) = \frac{1}{2}a^2$ to parameterize the classical moduli space. In quantum field theory, we parametrize the moduli space by the corresponding vacuum expectation values. Thus the moduli space of the $SU(2)$ $N = 2$ supersymmetric Yang-Mills theory is parametrized by $u =< \mathrm{Tr}(\phi^2) >$, which at the classical limit, reduces to $a^2/2$. Now the gauge group $SU(2)$ will be broken to $U(1)$. And the $U(1)$ R-symmetry will be break to $\mathbf{Z_8}$. Since u has R-charge 4, Under $\mathbf{Z}_8$, u will be transform to $e^{2\pi in/2}u$ with n take from 0 to 7, thus $\mathbf{Z}_8$ is break to $\mathbf{Z}_4$. The broken part $\mathbf{Z_2}$ will relate u to $-u$.

The $SU(2)$ gauge symmetry is spontaneously broken to $U(1)$ by the non-zero $\langle\phi\rangle$, In low energy region, we can integrate out all the massive modes to obtain a effective description of the theory in terms of massless modes. In practice, this procedure is not easy to implement. Seiberg and Witten found an indirect method to determine the exact low-energy theory.

In terms of massless chiral and vector superfields, we have effective low-energy Lagrangian

$$\mathcal{L}_{eff} = \frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta\, \frac{\partial\mathcal{F}(A)}{\partial A}\bar{A} + \int d^2\theta\, \frac{1}{2}\frac{\partial^2\mathcal{F}}{\partial A^2}W^\alpha W_\alpha\right] , \tag{8}$$

From this we get Kahler potential

$$K = \mathrm{Im}\left(\bar{A}\frac{\partial\mathcal{F}(A)}{\partial A}\right)$$

and the metric on the field space is given by

$$ds^2 = \mathrm{Im}(\tau)\, da d\bar{a}\,, \quad \text{with,} \quad \tau(a) = \frac{\partial^2\mathcal{F}}{\partial a^2}. \tag{9}$$

here a denotes the scalar component of the chiral superfield A Our tusk now is to determine the prepotential.

Since the theory is asymptotically free therefore at high energies one can determine the one-loop perturbative correction to $\mathcal{F}$. the $U(1)_{\mathcal{R}}$ symmetry of this theory is broken by the standard chiral anomaly

$$\partial_\mu J^\mu_R = -\frac{N_c}{8\pi^2} F_{\mu\nu}\widetilde{F}^{\mu\nu}\,.$$

This implies that, to one-loop, under a $U(1)_{\mathcal{R}}$ transformation, the effective Lagrangian changes by

$$\delta\mathcal{L}_{eff} = -\frac{\alpha N_c}{8\pi^2} F\widetilde{F}\,. \tag{10}$$

This will shift the θ-angle as $\theta \longrightarrow \theta + 4N_c\alpha$. Since θ has period 2π, we have $\alpha = 2\pi/4N_c$. Thus the anomaly breaks $U(1)_{\mathcal{R}}$ to $\mathbf{Z}_{4N_c}$.

The variation $\delta\mathcal{L}_{eff}$ could come only from terms in $\mathcal{L}_{eff}$ which are quadratic in $F_{\mu\nu}$. Writing only the relevent terms from the Lagrangian (8), this means

$$\delta\mathcal{L}_{eff} = \frac{1}{16\pi}\mathrm{Im}\left[(\mathcal{F}''(e^{2i\alpha}A) - \mathcal{F}''(A))(-FF + iF\widetilde{F})\right]$$

For infinitesimal α, we have

$$\frac{\partial^3\mathcal{F}}{\partial A^3} = \frac{N_c}{\pi}\frac{i}{A}$$

Integrate this, we get the one-loop prepotential as

$$\mathcal{F}_{1-loop}(A) = \frac{i}{2\pi}A^2\ln\frac{A^2}{\Lambda^2} \tag{11}$$

It is known that, due to $N = 2$ supersymmetry, the above one-loop expression for the prepotential does not receive higher order perturbative corrections. This is related to the fact that in this theory the perturbative β-function is only a one-loop effect.

The prepotential will receive non-perturbative corrections due to instanton effects [5]. The contribution of k-instanton should be proportional to the k-instanton factor $\exp(-8\pi^2 k/g^2)$. Using the one-loop β-function of the theory given by $\beta(g) = -N_c g^3/8\pi^2$, the k-instanton factor can be written as

$$e^{-8\pi^2 k/g^2} = \left(\frac{\Lambda}{a}\right)^{4k}\,.$$

From the Lagrangian(5), we note that the broken $U(1)_{\mathcal{R}}$ symmetry is restored if we assign a charge of 2 to Λ. With this modification, the prepotential should transform under $U(1)_{\mathcal{R}}$ as a field of charge 4, without a non-homogeneous term. This implies that the k-instanton correction should also be proportional to A^2. Thus the prepotential can be written as

$$\mathcal{F} = \frac{i}{2\pi}A^2\ln\frac{A^2}{\Lambda^2} + \sum_{k=1}^{\infty}\mathcal{F}_k\left(\frac{\Lambda}{A}\right)^{4k}A^2\,.$$

For large $|a|$, using (11), we have $\tau(a) = \frac{i}{\pi}(\ln\frac{a^2}{\Lambda^2} + 3)$. This is a multi-valued function. The metric on the field space given by $\mathrm{Im}\tau$ is single valued. since $\mathrm{Im}\tau(a)$ is a harmonic function, it cannot have a global minimum. Thus, if it is globally defined, it cannot be positive everywhere. Thus this description of metric will be valid only locally. In the region of the complex plane where $\tau(a)$ becomes negative, one needs a different description of the theory.

3 Duality

We find a dual description of original variable. In this description, the dual vector field is coupled to the monopoles as ordinary vector field coupled to electrons. Thus the duality transformation will exchange the elementary modes as electrons with collective modes as monopoles. To demonstrate this explicitly, we write down the bosonic part of the action

$$\frac{1}{32\pi}\,\mathrm{Im}\int \tau(a)(F+i\widetilde{F})^2=\frac{1}{16\pi}\,\mathrm{Im}\int \tau(a)(F^2+i\widetilde{F}F)$$

Now we regard F as an independent field and implement the Bianchi identity $dF=0$ by introducing a Lagrange multiplier vector field V_D. For one monopole, we have $\epsilon^{0\mu\nu\rho}\partial_\mu F_{\nu\rho}=8\pi\delta^{(3)}(x)$. If V_D couples to it as A_μ couples to j_μ, we have

$$\frac{1}{8\pi}\int V_{D\mu}\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}=\frac{1}{8\pi}\int \widetilde{F}_D F=\frac{1}{16\pi}\,\mathrm{Re}\int(\widetilde{F}_D-iF_D)(F+i\widetilde{F})$$

where, $F_{D\mu\nu}=\partial_\mu V_{D\nu}-\partial_\nu V_{D\mu}$. Adding this to the gauge field action and integrating over F, gives the dual theory

$$\frac{1}{32\pi}\,\mathrm{Im}\int\left(-\frac{1}{\tau}\right)(F_D+i\widetilde{F}_D)^2=\frac{1}{16\pi}\,\mathrm{Im}\int\left(-\frac{1}{\tau}\right)(F_D^2+i\widetilde{F}_D F_D)$$

We can do the same thing for the vector superfield. Now the Bianchi identity is $\mathrm{Im}\mathcal{D}W=0$. Similarly we have Lagrange multiplier term

$$\frac{1}{4\pi}\,\mathrm{Im}\int d^4x d^4\theta\, V_D\mathcal{D}W=\frac{1}{4\pi}\,\mathrm{Re}\int d^4xd^4\theta\, i\mathcal{D}V_D W=-\frac{1}{4\pi}\,\mathrm{Im}\int d^4xd^2\theta\, W_D W$$

Adding this to the action and integrating out W, gives the dual action

$$\frac{1}{8\pi}\,\mathrm{Im}\int d^2\theta\left(-\frac{1}{\tau(A)}W_D^2\right)$$

We can further transform chiral superfield A. Define $h(A)=\partial\mathcal{F}/\partial A$. The scalar kinetic energy term becomes $\mathrm{Im}\int d^4\theta\, h(A)\bar{A}$. By introducing transformation $A_D=h(A)$ a, It will become $\mathrm{Im}\int d^4\theta\, h_D(A_D)\bar{A}_D$.

From above we found the effective coupling transform as

$$\tau\to\tau_D=-\frac{1}{\tau}\,. \tag{12}$$

This is the electric-magnetic duality. And the action is also invariant under the replacement $\tau\to\tau+1$. These transformation can be implemented by the matrices

$$S=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad T=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}. \tag{13}$$

They generate the $SL(2,\mathrm{Z})$ group. This the most general way to transform our description of the metric. It will transform A and A_D as

$$\begin{pmatrix} A_D \\ A \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A_D \\ A \end{pmatrix} . \tag{14}$$

Let a_D to be the scalar component of A_D, In terms of this, the metric takes the form

$$ds^2 = \operatorname{Im} da_D d\bar{a} = -\frac{i}{2}(da_D d\bar{a} - da d\bar{a}_D)$$

which is manifestly invariant under the $SL(2,\mathrm{Z})$ transformations. The moduli space $\mathcal{M}$ is a complex one dimensional manifold parameterized by u . Let a and a_D be the coordinates on a space $X \cong \mathrm{C}^2$ on which we can choose a symplectic form $\omega = \operatorname{Im} da_D \wedge d\bar{a}$. The functions $(a_D(u), a(u))$ give a map from $\mathcal{M}$ to X. In other words, they determine a section of X regarded as an $SL(2,\mathrm{Z})$ bundle over $\mathcal{M}$.

The monopole is a static and finite energy semi-classical solution to the ordinary Yang-Mills- Higgs theory[4]. If we take the theta term into account, the magnetic charge will also contribute to electric charge, thus we will have dyons. Their mass satisfy the BPS bound condition $M \geq \sqrt{2}|Z|$, with

$$Z = a(n_e + \tau\, n_m)$$

The above Z can be showed to equal to the central charge of SUSY algebra. All states that saturate this bound is called BPS states. In this situation, we know half of the rising operator will act trivially as in the massless irreducible representation. They form a short multipet of $N = 2$ SUSY algebra. They are the hypermultiplets we discuss above, which in the $N = 1$ formalism are described by chiral fields $M, \widetilde{M}$. If a hypermultiplet carries charge n_e, then its coupling to the chiral field A is uniquely fixed by $N = 2$ supersymmetry as

$$\sqrt{2}\, n_e\, A M \widetilde{M}$$

From this we deduce $Z = a\, n_e$ for such a state. We can do the same thing in dual description for a monopole with magnetic charge n_m,we have $Z = a\, n_m$. Thus for dyons we have

$$Z = a n_e + a_D n_m \, . \tag{15}$$

4 Singularity on the Moduli Space

As we discussed above, the metric of the moduli space can not be described by a or a_D globally. We can transform them under $SL(2,Z)$ group to make the metric positive everywhere. From the perturbative form of a and a_D, we can see they are multiple valued functions. If the variable move around a branch singularity, they will transform to some linear combination of themselves. This transformation is called monodromy. We will expect different monodromy for different singularity. Together, they will from a subgroup of $SL(2,Z)$. Seiberg and Witten analyzed the singularity and monodromy of moduli space, and use these information to determine $a(u)$ and $a_D(u)$ exactly.

First, we can find a non-trivial monodromy at infinity. Near infinity, the theory is asymptotically free. We can use perturbative result $u = \frac{1}{2}a^2$ and $\mathcal{F}(a) = (i/2\pi)a^2 \ln(a^2/\Lambda^2)$. From this we get

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = \frac{2ia}{\pi} \ln(\frac{a}{\Lambda}) + \frac{ia}{\pi}$$

If we move around $u = 0$ on the u-plane, we get $\ln u \to \ln u + 2\pi i$, and $\ln a \to \ln a + i\pi$, hence,

$$\begin{aligned} a_D &\to -a_D + 2a \\ a &\to -a \end{aligned}$$

Corresponding to the monodromy matrix

$$M_\infty = PT^{-2} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \tag{16}$$

Under this monodromy, the magnetic and electric charge of BPS states will change as $(n_m, n_e) \to (-n_m, -n_e - 2n_m)$.

The singularity at infinity implies that there exist other singularities at finite u in order to make a branch cut. If their monodromies commute with M_∞, then a^2 is a good global coordinate. This contradict with our above discussion. To get a non-abelian monodromy group, we need at least two singularities at finite u, with non-trivial monodromies around them. These singularities will be related by the broken $\mathbf{Z_2}$ symmetry $u \to -u$. We make this minimal assumption on the number of extra singularities. Then a loop enclosing both these singularities should reproduce the monodromy M_∞. In the following, we assume two more singularities at $u = u_0$ and $u = -u_0$. Since moving around the two singularities is equivalent to moving around the infinity, we have $M_{u_0} M_{-u_0} = M_\infty$.

Physically, these singularities will correspond to extra massless particle. Since we obtain the low- energy effective theory by integrating out the massive modes. If at some point the massive modes become massless, then integrating out massless modes will generate singularities on the moduli space. If the massless particle is non-Abelian gauge boson, then we will expect comformal invariant in low energy limit. This will contradict with $u =< \mathrm{Tr}(\phi^2) >\neq 0$ unless $< \mathrm{Tr}(\phi^2) >$ has dimension zero. This means operator $\mathrm{Tr}(\phi^2)$ will be mixed with identity under renormalization. But we have showed u is odd under $\mathbf{Z_2}$ symmetry. The identity operator has no such property. Thus the mixing will lead to a confliction. The massless mode will not be gauge boson.

Thus we should focus on the massive particle with spin less than one half. In pure gauge theory, the only choice is that collective modes such as monopole or dyon that become massless at singularity points. In $N = 2$ SUSY, a massive multipet with spin $\leq 1/2$ must be a hypermultiplet, which belongs to a short representation. Thus monopole and dyon are BPS states and satisfy mass formula (15). If a monopole becomes massless, we have $a_D(u_0) = 0$. As before, we should transform to a dual theory to describe monopole. Now the low-energy theory looks like QED, thus the gauge coupling goes to zero as the mass of monopole goes to zero. Near u_0 we can use perturbation result. From the β-function of QED,

$$\beta(g) = \frac{g^3}{16\pi^2} \left(\sum_f \frac{2}{3} Q_f^2 + \sum_s \frac{1}{3} Q_s^2 \right) = \frac{g^3}{8\pi^2}$$

we get

$$\mu \frac{d\tau}{d\mu} = -\frac{i}{\pi} Q^2 \,.$$

Integrate to get

$$\tau_D \approx -\frac{i}{\pi} \ln a_D$$

Near u_0 we can expand as $a_D \approx c_0(u - u_0)$, Since $\tau_D = dh_D/da_D$ we can integrate to get

$$a(u) = -h_D(u) \approx a_0 + \frac{i}{\pi}\, c_0(u - u_0) \ln(u - u_0)$$

When we move around u_0, we have $\ln(u - u_0) \to \ln(u - u_0) + 2\pi i$, thus

$$\begin{array}{rcl} a_D & \to & a_D \\ a & \to & a - 2\, a_D \end{array} \tag{17}$$

In the following, we set $u_0 = 1$. Monodromy matrix is

$$M_1 = S T^2 S^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

Then for $u = -1$ we have

$$M_1 = (TS) T^2 (TS)^{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

If we denote charge vector $q = (n_m,\, n_e)$, then easy to see monopole $q_1 = (1,\, 0)$ is eigenvector of M_1, $q_1\, M_1 = q_1$. From M_{-1}, $q_{-1}\, M_{-1} = q_{-1}$, we get $q_{-1} = (1,\, -1)$, thus the massless particle is a dyon.

Now the moduli space$\mathcal{M}$ is the u-plane with the above three singularity and $\mathbf{Z_2}$ symmetry relating u to $-u$. Over this punctured plane there is a flat $SL(2, \mathbf{Z})$ bundle V, which has $(a_D, a)^T$ as a section. This bundle has monodromies around singularities as discussed above. he monodromies generate a subgroup $\Gamma(2)$ of $SL(2, \mathrm{Z})$.

From these, Seiberg and Witten invent a method to determine $a_D(u)$ and $a(u)$, thus also determine the metric on the moduli space. This method involve some mathematical background about complex curve and Riemann surface. Basic idea is to construct a family of curves whose moduli space is just $\mathcal{M}$.

In our situation, we consider the family of curves E_u described by the equation

$$y^2 = (x - 1)(x + 1)(x - u)\,, \tag{18}$$

Note that this equation is invariant under the transformations

$$w : \{u \to -u\,, x \to -x\,, y \to \pm i y\}\,,$$

that generate a $\mathbf{Z}_4$ symmetry, out of which only a $\mathbf{Z}_2$ subgroup acts on u. This is the same as the symmetry structure on $\mathcal{M}$. For fixed u, the curve is parameterized by the x-space, which is two complex plane joined together through two cuts: from -1 to 1 and from u to ∞. If we include infinite point in the complex plane, it is equivalent to a sphere. And the two cuts can be treat as two tubes connection the two spheres. Thus the x-space is equivalent to a genus one Riemann surface or a torus. For different u, we get different torus. There two type of cycles on the torus, a-cycle around the cross section, b-cycle along the torus. T When $u = 1, -1, \infty$, the two branch points of x-space coincide and the torus will develop vanishing cycle, which correspond to the singularity on u-plane.

The torus E_u has a two dimensional first homology group $V_u = H^1(E_u, \mathrm{C})$. It can be think as fibres of a flat bundle over u-plane, and this bundle as section a and a_D. We can pick one-cycles γ_1 and γ_2 to form a basis for the first homology group V_u. On Riemann surface we can define differential one-form. The closed one-form modulo exact forms will form the first cohomology group, which is dual vector space of first homology group. Thus we can also regard λ as an element in V_u. We can choose a basis $\lambda_1 = dx/y$, $\lambda_1 = xdx/y$. λ_1 is also the unique holomorphic one on E_u. The period of the torus is defined by

$$b_i = \oint_{\gamma_i} \lambda_1 \,, \qquad \text{for} \quad i = 1, 2$$

then the torus is characterized by a parameter

$$\tau_u = b_1/b_2 \,, \qquad \text{with} \quad \mathrm{Im}(\tau_u) > 0$$

We want to find a one-form $\lambda = a_1\lambda_1 + a_2\lambda_2$ in V_u that satisfy

$$a_D = \oint_{\gamma_1} \lambda \,, \qquad a = \oint_{\gamma_2} \lambda$$

Suppose that

$$\frac{d\lambda}{du} = f(u)\lambda_1 = f(u)\frac{dx}{y} \,.$$

Then,

$$\frac{da_D}{du} = f(u)\, b_1 \,, \qquad \frac{da}{du} = f(u)\, b_2 \,,$$

so that

$$\tau = \frac{b_1}{b_2} = \tau_u \,.$$

Since $\mathrm{Im}\,\tau_u > 0$, we get $\mathrm{Im}\,\tau > 0$. The converse is also true, so $d\lambda/du$ does not depend on λ_2. The function $f(u)$ is fixed by the asymptotic behaviour of the theory near the singularities on the u plane and is given by $f(u) = -\sqrt{2}/4\pi$. With this, we can obtain

$$a(u) = \frac{\sqrt{2}}{\pi} \int_{-1}^{1} \frac{dx\sqrt{x-u}}{\sqrt{x^2-1}} \,. \tag{19}$$

For γ_1, we choose the curve which loops around the points 1 and u and get

$$a_D(u) = \frac{\sqrt{2}}{\pi} \int_{1}^{u} \frac{dx\sqrt{x-u}}{\sqrt{x^2-1}} \,. \tag{20}$$

It can be checked that they have desired behavior near singularities.

5 $N = 2$ supersymmetric gauge theory with matter

Seiberg and Witten also generalized their theory to include matter. Because more elements are involved here, their analysis is much more technically complicated than before. Thus we can only briefly discuss them.

Now we consider the $N = 2$ supersymmetric gauge theory, with gauge group $SU(2)$, coupled to N_f matter multiplets. As in usual QCD, matter fields (fermions) and gauge bosons contribute to the β-function with opposite signs. Thus, to insure asymptotic freedom, we require $N_f \leq 4$. The theory with $N_f = 4$ is finite to all orders in perturbation theory and, when the quarks are massless, it is very likely conformally invariant non-perturbatively. Thus $N_f = 4$ case is totally different from they others. In the following we will focus on $N_f = 1, 2, 3$.

In our previous convention, when quarks are present, n_e could be half-integral. To ensure that n_e is always integer, we multiply it by 2 and divide a by 2 to keep the mass formula unchanged. Since a_D is kept unchanged, it is now given by $2a_D = \partial\mathcal{F}/\partial a$. In terms of the gauge invariant quantity $u = \mathrm{Tr}\phi^2$, as $|u| \to \infty$, the asymptotic behavior of a and a_D now becomes

$$a \cong \frac{1}{2}\sqrt{2u}\,, \quad a_D \simeq i\frac{4}{\pi}a\log a$$

And the effective coupling is also rescaled to $\tau = \partial a_D/\partial a = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$. This rescaling of the charge affects the form of the curve which determines the solution of the model, keeping the physics unchanged.

We first consider the global symmetry of the theory and classical moduli space.The $N = 2$ supersymmetric QCD coupled to N_f matter hypermultiplets contains the $N = 1$ superpotential

$$W = \sum_{i=1}^{N_f}(\sqrt{2}\widetilde{Q}_i\Phi Q^i + m_i\tilde{Q}_iQ^i)\,.$$

In general, when $m_i = 0$, this theory has a global symmetry group $SU(N_f)\times SU(2)_R\times U(1)_{\mathcal{R}}$. In the special case, when the gauge group is $SU(2)$, the flavour group is enlarged to $O(2N_f)$. This is due to the fact that for $SU(2)$ the fundamental representation and its conjugate are isomorphic and, therefore, Q^i and $\widetilde{Q}_i$ can be combined into a $2N_f$-dimensional vector transforming under $O(2N_f)$. Thus, for the special case of the $SU(2)$ gauge group, the theory also has a parity symmetry group $\mathbf{Z}_2 \subset O(2N_f)$ acting as $\rho : Q_1 \leftrightarrow \widetilde{Q}_1$ with all other fields remaining unchanged. This special parity is important for the following discussion.

Classically, there is always a flat direction with non zero ϕ. Along this direction the gauge symmetry is broken to $U(1)$ and all quarks are massive. we call this branch of moduli space as the Coulomb branch.For $N_f = 0, 1$, this is the only flat direction. For $N_f \geq 2$, we can have new flat direction for non zero Q^i. This will completely break gauge symmetry, and we call it Higgs Branch. Below we will focus on Coulomb branch.

5.1 Some Property of Quantum Theory

In quantum theory, $U(1)_{\mathcal{R}}$ symmetry will be anomalous. The anomaly has the same form but with N_c replaced by $N_c - N_f/2$. Thus $U(1)_{\mathcal{R}}$ symmetry breaks to $\mathbf{Z}_{8-2N_f}$. This $\mathbf{Z}_{8-2N_f}$ can be combined with parity ρ, to form a anomaly-free group $\mathbf{Z}_{4(4-N_f)}$. That is because physical variables like Green function will at most change sign under $\mathbf{Z}_{4(4-N_f)}$, we can use above parity to change the sign back. Under this transformation, the gauge invariant order parameter $u = \mathrm{Tr}\phi^2$ has charge 4 and transforms as $u \to \exp 2\pi i/(4-N_f)u$. This further breaks $\mathbf{Z}_{4(4-N_f)}$ down to $\mathbf{Z}_4$. The remaining $\mathbf{Z}_{4-N_f}$ acts non-trivially on the u-plane. Thus the u-plane has global symmetry $\mathbf{Z}_{4-N_f}$ for $N_f > 0$.

As in the $N_f = 0$ case, the large u-behaviour of $a_D(u)$ can be obtained from the anomaly mentioned above. The result is

$$\begin{aligned} a &\cong \frac{1}{2}\sqrt{2u} \\ a_D &\simeq i\,\frac{4-N_f}{2\pi}\,a(u)\ln\frac{u}{\Lambda_{N_f}^2} \end{aligned} \tag{21}$$

The should also recieve contributions from instantons. Now the β has the same form as before, only with N_c replaced by $N_c - N_f/2$. Thus k-instanton contribution is proportional to the k-instanton factor, which, using the β-function, can be written as

$$e^{-8\pi^2 k/g^2} = \left(\frac{\Lambda_{N_f}}{a}\right)^{k(4-N_f)} . \tag{22}$$

If we assign charge 4 and even ρ-parity to u and charge $2(4-N_f)$ and odd ρ-parity to $\Lambda_{N_f}^{4-N_f}$, $U(1)_{\mathcal{R}}$ and ρ parity symmetry can be restored. Since the metric on the u-plane is invariant under the ρ-parity, odd instanton numbers cannot contribute to a and a_D. Putting these facts together, we can write the generic form a and a_D as

$$\begin{aligned} a &= \frac{1}{2}\sqrt{2u}\left(1 + \sum_{n=1}^{\infty} a_n(N_f)\left(\frac{\Lambda_{N_f}^2}{u}\right)^{n(4-N_f)}\right) , \\ a_D &= i\frac{4-N_f}{2\pi}a(u)\ln\frac{u}{\Lambda_{N_f}^2} + \sqrt{u}\sum_{n=0}^{\infty} a_{Dn}(N_f)\left(\frac{\Lambda_{N_f}^2}{u}\right)^{n(4-N_f)} . \end{aligned}$$

Under the $\mathbf{Z}_{4-N_f}$ symmetry of the u plane, a transforms linearly. From the first term of a_D, we can see a_D will get a term proportional to a under this symmetry. This means that this symmetry will shift the electric charge of monopoles.

5.2 BPS States

On the Higgs branch $SU(2)$, is completely broken and there are no electric or magnetic charges. Thus the central charge of the $N = 2$ algebra only contains contributions from the $U(1)$ charges of the hypermultiplets.

On the Coulomb branch, there are two kinds of BPS states. The simplest BPS saturated states are the quarks with zero bare mass and which acquire masses $M = \sqrt{2}|a|$ after the spontaneous breaking of the gauge symmetry. These form a set of BPS states which transform as a vector of $SO(2N_f)$. Besides these, there are monopoles which transform as a spinor of $SO(2N_f)$. In the presence of a monopole each $SU(2)$ doublet of fermions has one zero-mode. Since there are N_f hypermultiplets, there are $2N_f$ fermion doublets and, therefore, $2N_f$ fermion zero-modes. After quantization, these zero-modes give rise to a $2N_f$-dimensional Dirac algebra which provides a spinor representation of $SO(2N_f)$. Thus the fermion zero-modes turn the monopole into a spinor of $SO(2N_f)$. The presence of spinors indicate that, at the quantum level, the symmetry group is a universal cover of $SO(2N_f)$, or $Spin(2N_f)$.

The spectrum may also include states with $n_m \geq 2$. They can be interpreted as bound states of known particles. For $N_f \geq 2$, A convenient way of labelling states is in terms of their behaviour under the centre of $Spin(2N_f)$. For $N_f = 2$, the universal cover is $SU(2) \times SU(2)$ and the center is $\mathbf{Z}_2 \times \mathbf{Z}_2$. We write a representation of $\mathbf{Z}_2 \times \mathbf{Z}_2$ as (ϵ, ϵ'), where $\epsilon = 0$ for the trivial representation of $\mathbf{Z}_2$ and $\epsilon = 1$ for the non-trivial representation. Thus vectors transform as $(1, 1)$, while spinors transform as $(1, 0)$. Then states of (n_m, n_e) will transform under the center as $((n_e + n_m)\ mod\ 2, n_e\ mod\ 2)$. For $N_f = 3$, the universal cover of $SO(6)$ is $SU(4)$, and its center is $\mathbf{Z}_4$. vectors transform has charge 2, while spinors has charge 1. Thus state of (n_m, n_e) will transform under the center as $\exp(\frac{i\pi}{2}(n_m + 2n_e))$.

If an $N = 2$ invariant mass is turned on, say $m_{N_f} \neq 0$, then $SO(2N_f)$ explicitly breaks to $SO(2N_f - 2) \times SO(2)$, and the global Abelian charge associated with $SO(2)$ appears in the central charge

$$Z = n_e a + n_m a_D + S_{N_f} \frac{m_{N_f}}{\sqrt{2}}, \qquad M = \sqrt{2}|Z|\,.$$

Hence for $a = \pm m_{N_f}/\sqrt{2}$ one of the elementary quarks becomes massless.

5.3 Duality

As before, we can compute the monodromy matrices for (a_D, a) around the singular points on the moduli space. If we have a single quark with non-zero bare mass, then when $a \approx a_0 \equiv m_{N_f}/\sqrt{2}$ one of the quark will become massless. Near the singularity we have

$$\begin{aligned} a &\approx a_0\,, \\ a_D &\approx -\tfrac{i}{2\pi}(a - a_0)\ln(a - a_0) + c\,. \end{aligned}$$

The monodromy can now be easily computed to be

$$\begin{aligned} a &\rightarrow a\,, \\ a_D &\rightarrow a_D + a - \frac{m_{N_f}}{\sqrt{2}}\,. \end{aligned}$$

Unlike the $N_f = 0$ case, now the the monodromy transformation of $(a_D, a)^T$ has a inhomogeneous term besides the usual $SL(2, Z)$. In the pure gauge theory case, the shift was not compatible with the symmetries of the BPS mass formula. In the presence of matter, this shift is allowed since the BPS mass formula is modified. Now the shift naturally appears as a part of the monodromy group. To write a monodromy matrix, we construct a column vector $(m/\sqrt{2}, a_D, a)^T$. The monodromy can now be written as

$$\begin{pmatrix} m/\sqrt{2} \\ a_D \\ a \end{pmatrix} \rightarrow \mathcal{M} \begin{pmatrix} m/\sqrt{2} \\ a_D \\ a \end{pmatrix},$$

with the monodromy matrix $\mathcal{M}$ given by

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

If we arrange the charges as a row vector $W = (S, n_m, n_e)$, then invariance of Z (or of M) implies that, under the monodromy, $W \rightarrow W\mathcal{M}^{-1}$. In general the matrix $\mathcal{M}$ can be of the form

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ r & k & l \\ q & n & p \end{pmatrix}, \qquad \mathcal{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ lq - pr & p & -l \\ nr - kq & -n & k \end{pmatrix},$$

with $\det \mathcal{M} = 1$.

Note that under the monodromy, the electric and magnetic charges can mix with each other but will not pick up contributions proportional to S which is a global symmetry charge. On the other hand, S can pick up contributions proportional to n_e and n_m which are related to the local gauge symmetry.

5.4 Singularities

To determine the singularity structure on the moduli space, we first consider theories with $N_f \leq 3$, and with hypermultiplet bare masses very large as compared to Λ. The singularities which arise from hypermultiplets becoming massless are now in the semi-classical (large u) region of the moduli space and can be easily identified. In the small u region, one is effectively left with an $N_f = 0$ theory with two singularities corresponding to massless monopoles and dyons. Then we slowly decrease the bare masses to zero and follow the movement of the sigularities on the moduli space.

For $N_f = 3$, Let us start with equal masses $m_i = m >> \Lambda$. In this case, the global symmetry $Spin(6) \approx SU(4)$ of the massless theory is broken to $SU(3) \times U(1)$. Classically, there is a singularity at $a = m/\sqrt{2}$ where the three quarks become massless. These electrically charged massless fields form a **3** representation of $SU(3)$. For $m >> \Lambda$ the singularity is in the semiclassical region, $u \sim 2a^2 >> \Lambda^2$, thus it will persists quantum mechanically. For $u << m^2$ the three massive quarks can be integrated out giving a $N_f = 0$ theory with scale $\Lambda_0^4 = m^3\Lambda_3$. Hence, for small u the moduli space is that of a pure $N = 2$ theory with scale Λ_0 which has two singularities with $(n_m, n_e) = (1, 0)$ and $(1, 1)$. These are the points where monopoles and dyons become massless. Clearly the massless states at these singularities are $SU(3)$ invariant.

As m is decreased, the singularity at large u moves, and although the Abelian charges such has n_e or n_m can change, their non-Abelian charges cannot change. Hence, for any m, the massless fields at the various singularities transform as $\mathbf{3}$, $\mathbf{1}$ and $\mathbf{1}$ of $SU(3)$. In the $m = 0$ limit, the original global symmetry is restored and the massless states must combine into representations of $SU(4)$. The only possibility is to have two singularities combining into a $\mathbf{4}$ of $SU(4)$ and the other sigularity moves somewhere else remaining a singlet. Thus, we conclude that the massless $N_f = 3$ theory has two singularities with massless particles in the $\mathbf{4}$ and $\mathbf{1}$ of $SU(4)$. From our study of how different states transform under the centre, the smallest charges for the massless particles at the singularities are $(n_m, n_e) = (1, 0)$ for the $\mathbf{4}$, and $(n_m, n_e) = (2, 1)$ for the $\mathbf{1}$.

The same procedure can be used to determine the singularities of the massless $N_f = 1, 2$ theories. They can also be determined by make some quark very heavy in $N_f = 3$ theory.First we give a mass m_3 to only one of the hypermultiplets. It breaks $SO(6) \rightarrow SO(2) \times SO(4)$ (or equivalently $SU(4) \rightarrow SU(2) \times SU(2) \times U(1)$). The four monopoles are split to two pairs. One pair transforms as $(\mathbf{2}, \mathbf{1})$ and the other as $(\mathbf{1}, \mathbf{2})$ under the unbroken $SU(2) \times SU(2) \times U(1)$. Therefore, for small non-zero m_3 there are three singularities. At two of them there are two massless particles transforming as one or the other spinor of $SO(4)$, while at the third singularity there is a massless $SO(4)$ singlet state. Now we increase m_3. For large m_3, we expect one singularity at large u where the elementary quark becomes massless. This state is an $SO(4)$ singlet, thus we conclude it is the third singularity we mentioned before.

We can now integrate out the heavy quark, eliminating the singularity just described (since it goes to infinity for $m_3 \rightarrow \infty$) and leaving the other singularities. The low energy theory is the massless $N_f = 2$ theory. Its scale Λ_2 is determined by a one loop calculation to be $\Lambda_2^2 = m_3\Lambda_3$. As we take m_3 to infinity holding Λ_2 fixed we are left with the two other singularities of the $(m_1 = m_2 = 0, m_3 \neq 0)$ $N_f = 3$ theory. Each of those singularities has two massless states in one or the other spinor representation of $SO(4)$. The minimal choices for their electric and magnetic charges are $(n_m, n_e) = (1, 0)$ and $(n_m, n_e) = (1, 1)$.

We can do the same thing to $N_f = 1$ theory. A mass term for the second quark m_2 breaks $SO(4) \rightarrow SO(2) \times SO(2)$. This will split the two pair of monopoles into four singlets. Thus the massive theory has four singularities. As m_2 becomes large, one of these singularities moves to large u as before. Three singularities are left behind. Now we integrate out the massive quark. The low energy theory is the massless $N_f = 1$ theory with scale $\Lambda_1^3 = m_2\Lambda_2^2$. This theory has the three singularities that do not go to infinity as $m_2 \rightarrow \infty$. They are related by the discrete $\mathbf{Z}_3$ global symmetry of the $N_f = 1$ theory. As we move from one singularity to the other, the monopole acquires one unit of electric charge. Therefore, the values of (n_m, n_e) for the massless states at the singularities are $(1, 0)$, $(1, 1)$ and $(1, 2)$.

From the charge spectrum of the particles which become massless at the singularities on the u-plane, we can get the corresponding monodromies. As in the $N_f = 0$ theory, the monodromy at $u = \infty$ can be obtained form the large u behavior of a and a_D,

$$\mathcal{M}_\infty = PT^{N_f - 4}$$

The singularities at finite points in the u-plane, in general, correspond to monopoles or dyons. To calculate the corresponding monodromy, we have to go to a dual description of the theory. In this frame, the monodromy is determined by the one-loop QED β-function with k hypermultiplets and is given by T^k. This has to be conjugated with $T^{n_e}S$ which converts a hypermultiplet of charge $(0,1)$ into a monopole of charge $(1,n_e)$. Hence the monodromy around a point with k magnetic monopoles with $(n_m=1,n_e)$ is $(T^{n_e}S)T^k(T^{n_e}S)^{-1}$. Similar arguments can be applied to calculate the monodromy for the $(2,1)$ state in the $N_f=3$ theory. In the following we list all the monodromies around the singularities described in the previous subsection:

$$\begin{array}{ll} N_f=1: & STS^{-1},\, (TS)T(TS)^{-1},\, (T^2S)T(T^2S)^{-1} \to M_\infty = PT^{-3}\,, \\ N_f=2: & ST^2S^{-1},\, (TS)T^2(TS)^{-1} \to M_\infty = PT^{-2}\,, \\ N_f=3: & (ST^2S)T(ST^2S)^{-1},\, ST^4S^{-1} \to M_\infty = PT^{-1}\,. \end{array}$$

From these monodromies, we can construct elliptic curves as we did in pure gauge theory. Then the moduli space of vacuum will be equivalent to the moduli space of these curves. Then a and a_D is also determine by the one cycle integral of certain one form.

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