Addendum

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This addendum is an effort to say something about the quantum aspects of the supersymmetric nonlinear sigma models, and at the same time, show more explicitly how the $d = 2$ nonlinear sigma model is a fundamental concept and tool in string theory.

1 The Bosonic String, Nonlinear Sigma Models, and β-functions

Let us warm up by considering the (non-supersymmetric) bosonic string. (This discussion follows section 3.4 of [1]). The action for the bosonic string propagating in flat Minkowski space is

$$\mathcal{S} = -\frac{1}{4\pi\alpha}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}(X^\rho), \tag{1}$$

where $h_{\alpha\beta}$ is the metric of the string worldsheet, (σ^1, σ^2) are the coordinates on the worldsheet, and $X^\mu(\sigma)$ is the position of the string in the space-time. For the moment, α can be viewed as a perturbation series expansion parameter, playing the role of $\hbar$. It's significance will become more apparent below. The space-time dimension must be 26 to have a consistent quantum theory, in that there are no negative norm states. If one wishes to consider a string propagating, not in flat space, but in a curved space-time, the obvious generalization is

$$\mathcal{S} = -\frac{1}{4\pi\alpha}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X^\rho). \tag{2}$$

If one chooses conformal coordinates for the string worldsheet, $h_{\alpha\beta} = e^{\phi(\sigma)}\eta_{\alpha\beta}$, then the terms in (2) involving $h_{\alpha\beta}$ drop and one is left with

$$\mathcal{S} = -\frac{1}{4\pi\alpha}\int d^2\sigma\partial_\alpha X^\mu\partial^\alpha X^\nu g_{\mu\nu}(X^\rho). \tag{3}$$

Observe that this is a nonlinear sigma model! The fields are maps from the world-sheet of the string to the space-time. So here, the target manifold M is actually the space-time.

Now, the convenient choice of conformal coordinates made above, does not come without a price. By choosing conformal coordinates one is making a choice of gauge. Therefore, the gauge-fixed action must be supplemented by appropriate gauge-fixing conditions. It turns out that the correct condition is the vanishing of the energy-momentum tensor,

$$T_{\alpha\beta} = 0. \tag{4}$$

Classically, there is no problem imposing such a constraint. However, in the quantum theory, anomalies might prevent one from maintaining the condition. Alternatively, that fact that all world-sheet metric dependance dropped out of (2) can be attributed to the scale invariance, or Weyl invariance, of the action. This invariance may break down in the quantum theory. It turns out that the breakdown of Weyl invariance is intimately related to the β-function of the nonlinear sigma model. Weyl invariance implies global scale invariance, which in turn implies the vanishing of the β-function and thus ultra-violet finiteness (modulo a possible wave function renormalization).

To see whether or not Weyl invariance breaks down, one may do a one-loop computation in $2+\epsilon$ dimensions. In $2+\epsilon$ dimensions, in conformal coordinates, the action takes the form

$$\tilde{\mathcal{S}} = -\frac{1}{4\pi\alpha}\int d^{2+\epsilon}\sigma e^{\epsilon\phi}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X^\rho). \tag{5}$$

The idea is to see if the ϕ dependance drops out as one takes the limit $\epsilon \to 0$. Now two expansions are made. The first is $e^{\epsilon\phi} = 1 + \epsilon\phi + \ldots$. The second involves an expansion of the metric $g_{\mu\nu}$. Around any point X_0^ρ there exist coordinates called Riemann normal coordinates which put the metric in the form $g_{\mu\nu}(X^\rho) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\lambda\ \nu\kappa}(X_0^\rho)x^\lambda x^\kappa + O((x^\mu)^4)$, where $x^\mu = X^\mu - X_0^\mu$. Using these expansions, (5) becomes

$$\tilde{\mathcal{S}} = -\frac{1}{4\pi\alpha}\int d^{2+\epsilon}\sigma[(\partial_\alpha x^\mu \partial^\alpha x^\nu)(1+\epsilon\phi)\eta_{\mu\nu} - \frac{1}{3}\partial_\alpha x^\mu \partial^\alpha x^\nu x^\lambda x^\kappa R_{\mu\lambda\nu\kappa}(X_0^\rho)(1+\epsilon\phi) + O((x^\rho)^5)]. \tag{6}$$

The ϕ dependance in the kinetic term can be related to a wave function renormalization. To extract all of the ϕ dependance associated with a possible one-loop contribution to the β-function, it is convenient to make a field redefinition of the form $x^\mu = (1 - \frac{1}{2}\epsilon\phi)y^\mu$. Then (6) becomes

$$\tilde{\mathcal{S}} = -\frac{1}{4\pi\alpha}\int d^{2+\epsilon}\sigma[(\partial_\alpha y_\mu \partial^\alpha y^\nu) + \frac{\epsilon}{2}y^2\partial^2\phi - \frac{1}{3}(1+\epsilon\phi)\partial_\alpha y^\mu \partial^\alpha y^\nu y^\lambda y^\kappa R_{\mu\lambda\nu\kappa} + O((y^\rho)^5)], \tag{7}$$

where the symmetry properties of $R_{\mu\nu\lambda\kappa}$ have been used. There are two logarithmically divergent one-loop Feynman diagrams coming from the last term, which give $1/\epsilon$ poles. They come from the $\langle y^\lambda y^\kappa \rangle$ contraction and the $\langle \partial_\alpha y^\mu \partial^\alpha y^\nu \rangle$ contraction. Hence, as one takes the limit $\epsilon \to 0$, there is a surviving ϕ dependance given by

$$\tilde{\mathcal{S}} = -\frac{1}{4\pi\alpha}\int d^2\sigma\phi[R_{\mu\nu}\partial_\alpha y^\mu \partial^\alpha y^\nu + 2 - loop]. \tag{8}$$

If, instead, one uses the Riemann normal coordinate expansion in the original action (3) to give

$$\mathcal{S}' = -\frac{1}{4\pi\alpha}\int d^2\sigma(\partial_\alpha x^\mu \partial^\alpha x^\nu - \frac{1}{3}\partial_\alpha x^\mu \partial^\alpha x^\nu x^\lambda x^\kappa R_{\mu\lambda\nu\kappa}(X_0^\rho) + O((x^\rho)^5)), \tag{9}$$

and computes the logarithmically divergent one-loop diagrams coming from the second term, one finds that the infinite term in the one-loop effective action given by

$$\Delta\mathcal{S} = -\frac{1}{24\pi\epsilon\alpha}\int d^2\sigma R_{\mu\nu}(X_0^\rho)\partial_\alpha X^\mu \partial^\alpha X^\nu. \tag{10}$$

From this, one extracts the one-loop β-functional

$$\beta_{\mu\nu}(X^\rho) = -\frac{1}{4\pi}R_{\mu\nu}(X^\rho). \tag{11}$$

Hence, on comparing (11) and (8), one finds that the β-function exactly measures the failure of Weyl invariance.

A consistent string theory must have conformal, or Weyl, invariance. Therefore, one is led to the conclusion the the β-function must vanish, and hence, the Ricci curvature must vanish $R_{\mu\nu} = 0$. Is it a mere coincidence that Weyl invariance at one-loop order implies Einstein's equation in vacuum? No. This is essentially an internal consistency check in the theory. While fascinating, it is not the direction in which this paper is heading. The interested reader is referred to [1].

The point of the above calculation was that the one loop β-function of the nonlinear sigma model (3) is proportional to the Ricci tensor of the target manifold. If one demands that the β-function vanishes, then a Ricci-flat target manifold is required. However, this is not entirely correct...

What is the correct interpretation of the action (3) for a string propagating in a curved space-time? It should really be viewed as the interaction the the string with gravity. Since gravitons are one of the massless modes of the bosonic string, this action is describing the interaction of the string with a background field of strings. But, there are other massless modes of the bosonic string. Besides the graviton there is an antisymmetric tensor mode $B_{\mu\nu}(X^\rho)$ and a scalar mode $\Phi(X^\rho)$, known as the *dilaton*. These modes should also be included as part of the background field of (3). The proper term for $B_{\mu\nu}$ is easily guessed:

$$\mathcal{S}_B = -\frac{1}{4\pi\alpha}\int d^2\sigma \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X^\rho). \tag{12}$$

The correct term for the dilaton is not so easy to guess. It turns out that the correct expression is

$$\mathcal{S}_\Phi = \frac{1}{4\pi}\int d^2\sigma \sqrt{h}\Phi(X^\rho)R^{(2)}, \tag{13}$$

where $R^{(2)}$ is the curvature scalar *of the world-sheet*. Therefore, the combined action is

$$\mathcal{S} = -\frac{1}{4\pi\alpha}\int d^2\sigma[\partial_\alpha X^\mu \partial^\alpha X^\nu g_{\mu\nu}(X^\rho) + \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X^\rho) - \alpha\sqrt{h}\Phi(X^\rho)R^{(2)}]. \tag{14}$$

There are two things to note about the dilaton contribution. Firstly, it explicitly violates Weyl invariance due to the appearance of the string world-sheet metric. Secondly, it is at higher order in the perturbation expansion parameter α. These two facts are related, of course. As the dilaton term explicitly breaks Weyl invariance, it must be considered as a quantum correction, relative to $g_{\mu\nu}$ and $B_{\mu\nu}$. In fact, though there appears to be a one parameter family of bosonic string theories, parameterized by α, it is not the case. α can be absorbed into a redefinition of the vacuum expectation value of Φ. Hence, the bosonic string theory is a single theory that has at tree level a one-parameter family of vacuum states labeled by the arbitrary expectation value of the massless scalar field Φ.

Note that, if we were to consider only the classical terms of (14), it has *exactly the form of the nonlinear sigma models with torsion considered earlier*: $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$. In order for the new theory to be conformally invariant at one loop, the conditions on the curvature, torsion, and dilaton field turn out to be

$$\begin{aligned}
&R_{\mu\nu} = 2D_\mu D_\nu \Phi - \frac{1}{4}H_\mu^{\lambda\rho}H_{\nu\lambda\rho}, \\
&D_\lambda H^\lambda_{\mu\nu} = 2(D_\lambda \Phi)H^\lambda_{\mu\nu}, \\
&4D_\mu D^\mu \Phi = 4D_\mu \Phi D^\mu \Phi + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}.
\end{aligned} \tag{15}$$

2 The Superstring and Supersymmetry

The bosonic string theory described above has a number of shortcomings. The ground state is a tachyon and there are no fermions present in its excitations. In attempts to generalize the theory to include fermionic degrees of freedom, one is led naturally to supersymmetry. In fact, the correspondence above, between the bosonic string in a background field and nonlinear sigma models, holds in this case as well. The propagation of superstrings in a fixed background space-time M can be described by a $1+1$ dimensional supersymmetric nonlinear sigma model. Note that, though the supersymmetry on the string world-sheet is actually local (gauged), a convenient choice of gauge can be made, analogous to the conformal gauge above, such that the theory "looks" exactly like the supersymmetric nonlinear sigma models studied previously.

The supercharges on the $d = 2$ Minkowski world-sheet may be taken as Majorana-Weyl spinors. Then, as discussed previously, one has some freedom in choosing how to implement the supersymmetry. For instance, Green-Schwarz superstrings have type $(1, 1)$ supersymmetry on the world-sheet while the heterotic string has type $(1, 0)$ supersymmetry on the world-sheet. The nonlinear sigma models that describe the propagation of these strings in a background field have type $(1, 1)$ or type $(1, 0)$ supersymmetry correspondingly [2]. The theory still must have conformal invariance, and the analysis made above continues to hold.

Finally, one may be interested in putting extra global supersymmetry on the world sheet. Consider the compactification of the 10-dimensional space-time, required by superstring theory, to $M^4 \times K$ where M^4 is four-dimensional Minkowski space-time and K is a compact six dimensional manifold. Then the string propagation on K is described by a $1 + 1$ dimensional sigma model with target manifold K. It turns out that, for the theory to have at least one unbroken supersymmetry in the four dimensional M^4, the nonlinear sigma model on K must have two supersymmetries. For instance, in the heterotic case, the supersymmetry must be of type $(2, 0)$ [3].

Hence, one is very interested in knowing what types of extended supersymmetry are possible in nonlinear sigma models, and what implications the extended supersymmetry has for the geometry of the target manifold. As we showed in the paper, for a nonlinear sigma model to posess type $(2, 0)$ supersymmetry, in the torsion free case, the target manifold must be Kahler. Now combine this with the requirement, from conformal invariance, that the manifold be Ricci-flat (in the case of no torsion). Together, these make a very strong restriction; a Kahler manifold that is Ricci flat is known as a *Calabi-Yau* manifold. And if you've read *The Elegant Universe* you know those are important!

In closing, it is hoped that this addendum has both (1) motivated the study of supersymmetric nonlinear sigma models, and (2) patched up, at least somewhat, the gaping hole in the paper with regards to the quantum aspects of the theories.

References

[1] M. Green, J. Schwarz, and E. Witten, *Superstring Theory, vol. 1*, Cambridge Univ. Press, 1987.

[2] C. Hull, *Sigma Model Beta-Functions and String Compactifications*, Nuc. Phys. **B267** (1986) 266.

[3] C. Hull and E. Witten, *Supersymmetric Sigma Models and the Heterotic String*, Phys. Lett. **160B** (1985) 298.