$N = 2$ Supersymmetric Yang-Mills Theory

Hui Dai

March 11th, 2005

Introduction

$N = 2$ Supersymmetric Yang-Mills is such a rich subject that it is beyond our ability to cover all the stuff in this final project. To discuss as much and as in details as we can, we decided to split our work: I will concentrate on the $N = 2$ pure $SU(2)$ theory, while Yan is going to do some discussion on theories with matter.

The first part of this paper is primarily devoted to the construction of the $N = 2$ supersymmetric Lagrangians with both gauge multiplets and matter multiplets from the $N = 1$ superfields and Lagrangians. While in the second part we describe the Seiberg-Witten analysis of the $N = 2$ pure $SU(2)$ gauge theory. We discuss the parametrization of the moduli space and the breaking of R-symmetries, and then we show how the chiral $U(1)$ anomaly of the theory can be used to obtain the one-loop form of the low-energy effective action. The rest is devoted to finding the exact low-energy effective action by using duality and the singularity structure on the moduli space of the theory.

1 The $N = 2$ Supersymmetric Lagragian

1.1 Review of the $N = 1$ Supersymmetry Representations

The $N = 1$ superspace is obtained by adding four spinor degrees of freedom $\theta^\alpha, \bar{\theta}_{\dot{\alpha}}$ to the space-time coordinates x^μ. There are two major multiplets for this case:

Chiral (or Scalar) Multiplet: The $N = 1$ chiral multiplet is characterized by a superfield with $\bar{D}_{\dot{\alpha}}\Phi = 0$. Any chiral superfield can be expanded as

$$\begin{aligned} \Phi(y,\theta) &= A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \\ &= A(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu A - \frac{1}{4}\theta^2\bar{\theta}^2\Box A \ + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta\theta F(x) \end{aligned} \tag{1}$$

where $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, A and ψ are the fermionic and scalar components respectively and F is an auxiliary field required for the off-shell closure of the algebra. Any holomorphic function of chiral superfields is a chiral superfield:

$$\begin{aligned}\mathcal{W}(\Phi_i) &= \mathcal{W}(A_i + \sqrt{2}\theta\psi_i + \theta\theta F_i) \\ &= \mathcal{W}(A_i) + \frac{\partial\mathcal{W}}{\partial A_i}\sqrt{2}\theta\psi_i + \theta\theta\left(\frac{\partial\mathcal{W}}{\partial A_i}F_i - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial A_i A_j}\psi_i\psi_j\right).\end{aligned} \tag{2}$$

The most general $N = 1$ supersymmetric Lagrangian for the scalar multiplet is

$$\mathcal{L} = \int d^4\theta\, K(\Phi, \Phi^\dagger) + \int d^2\theta\mathcal{W}(\Phi) + \int d^2\bar{\theta}\bar{\mathcal{W}}(\Phi^\dagger).$$

where $K(\Phi, \Phi^\dagger)$ is the Kähler potential, and $\mathcal{W}$ is referred to as the superpotential. The metric on the field space is given by $g^{ij} = \partial^2 K/\partial A_i \partial A_j^\dagger$.

Vector Multiplet: This multiplet is represented by a superfield satisfying $V = V^\dagger$. By power series expansion in θ and $\bar{\theta}$, we have

$$\begin{aligned}V(x,\theta,\bar{\theta}) = \; & C + i\theta\chi - i\bar{\theta}\bar{\chi} + \tfrac{i}{2}\theta^2(M+iN) - \tfrac{i}{2}\bar{\theta}^2(M-iN) \\ & -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}(\bar{\lambda} + \tfrac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi) \\ & -i\bar{\theta}^2\theta(\lambda + \tfrac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}) + \tfrac{1}{2}\theta^2\bar{\theta}^2(D - \tfrac{1}{2}\Box C).\end{aligned}$$

where C, D, M, N and A_μ are all real. The Abelian gauge transformations are given by $V \to V + \Lambda + \Lambda^\dagger$, where Λ ($\Lambda^\dagger$) are chiral (antichiral) superfields. In the Wess-Zumino gauge, $C = M = N = \chi = 0$, so

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D.$$

Thus $V^2 = \frac{1}{2}A_\mu A^\mu\theta^2\bar{\theta}^2$ and $V^3 = 0$. Fixing the gauge breaks supersymmetry, but not the gauge symmetry of the Abelian gauge field A_μ. The Abelian field strength is defined by

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V, \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}D^2\bar{D}_{\dot{\alpha}}V.$$

W_α is a chiral superfield. It is gauge invariant, so we can compute it in the Wess-Zumino gauge:

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha, \tag{3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the familiar Abelian field strength tensor. Thus

$$W^\alpha W_\alpha\,|_{\theta\theta} = -2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + D^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}.$$

The usual Abelian supersymmetric Lagrangian is given by

$$\mathcal{L} = \frac{1}{4g^2}\left(\int d^2\theta\, W^\alpha W_\alpha + \int d^2\bar{\theta}\,\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right).$$

In the non-Abelian case, V belongs to the adjoint representation of the gauge group: $V = V_A T^A$, where $T^{A\dagger} = T^A$. The gauge transformations are now

$$e^{-2V} \to e^{-i\Lambda^\dagger} e^{-2V} e^{i\Lambda} \qquad \text{where } \Lambda = \Lambda_A T^A$$

The non-Abelian gauge field strength is defined by

$$W_\alpha = \frac{1}{8} \bar{D}^2 e^{2V} D_\alpha e^{-2V}$$

It transforms as

$$W_\alpha \to W'_\alpha = e^{-i\Lambda} W_\alpha e^{i\Lambda} \, .$$

In components, it takes the form

$$W_\alpha = T^a \left(-i\lambda^a_\alpha + \theta_\alpha D^a - \frac{i}{2} (\sigma^\mu \bar{\sigma}^\nu \theta)_\alpha F^a_{\mu\nu} + \theta^2 \sigma^\mu D_\mu \bar{\lambda}^a \right) , \tag{4}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$, and $D_\mu \bar{\lambda}^a = \partial_\mu \bar{\lambda}^a + f^{abc} A^b_\mu \bar{\lambda}^c$.

Using the normalization $\mathrm{Tr}(T^a T^b) = \delta^{ab}$, we have

$$\mathrm{Tr}(W^\alpha W_\alpha \mid_{\theta\theta}) = -2i\lambda^a \sigma^\mu D_\mu \bar{\lambda}^a + D^a D^a - \frac{1}{2} F^{a\mu\nu} F^a_{\mu\nu} + \frac{i}{4} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \, . \tag{5}$$

The usual non-Abelian supersymmetric Lagrangian is given by

$$\mathcal{L} = \frac{1}{4g^2} \mathrm{Tr} \left(\int d^2\theta \, W^\alpha W_\alpha + \int d^2\bar{\theta} \, \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \right) .$$

However, we are interested in the supersymmetric Lagrangian with a θ-term :

$$\begin{aligned} \mathcal{L} &= \frac{1}{8\pi} \mathrm{Im} \left(\tau \, \mathrm{Tr} \int d^2\theta \, W^\alpha W_\alpha \right) \\ &= -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} + \frac{1}{g^2} (\frac{1}{2} D^a D^a - i\lambda^a \sigma^\mu D_\mu \bar{\lambda}^a) , \end{aligned} \tag{6}$$

where $\tau = \theta/2\pi + 4\pi i/g^2$ can be regarded as a constant chiral superfield.

The General $N = 1$ Lagrangian: From the previous discussion, we are now ready to write down the full $N = 1$ supersymmetric Lagrangian:

$$\mathcal{L} = \frac{1}{8\pi} \, \mathrm{Im} \left(\tau \mathrm{Tr} \int d^2\theta \, W^\alpha W_\alpha \right) + \int d^2\theta d^2\bar{\theta} \, \Phi^\dagger e^{-2V} \Phi + \int d^2\theta \, \mathcal{W} + \int d^2\bar{\theta} \, \bar{\mathcal{W}} \, . \tag{7}$$

Each term is separately invariant, so the relative normalization between the scalar part and the Yang-Mills part is not fixed by $N = 1$ supersymmetry. We are free to change the relative normalization by rescaling the scalar multiplet Φ. In terms of component fields,

$$\begin{aligned} \mathcal{L} = &- \frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} - \frac{i}{g^2} \, \lambda^a \sigma^\mu D_\mu \bar{\lambda}^a + \frac{1}{2g^2} D^a D^a \\ &+ (\partial_\mu A - i A^a_\mu T^a A)^\dagger (\partial^\mu A - i A^{a\mu} T^a A) - i \, \bar{\psi} \bar{\sigma}^\mu (\partial_\mu \psi - i A^a_\mu T^a \psi) \\ &- D^a A^\dagger T^a A - i\sqrt{2} \, A^\dagger T^a \lambda^a \psi + i\sqrt{2} \, \bar{\psi} T^a A \bar{\lambda}^a + F^\dagger_i F_i \\ &+ \frac{\partial \mathcal{W}}{\partial A_i} F_i + \frac{\partial \bar{\mathcal{W}}}{\partial A^\dagger_i} F^\dagger_i - \frac{1}{2} \frac{\partial^2 \mathcal{W}}{\partial A_i \partial A_j} \psi_i \psi_j - \frac{1}{2} \frac{\partial^2 \bar{\mathcal{W}}}{\partial A^\dagger_i \partial A^\dagger_j} \bar{\psi}_i \bar{\psi}_j \, . \end{aligned} \tag{8}$$

1.2 The $N = 2$ Supersymmetric Lagrangian for Gauge Fields

The on-shell $N = 1$ scalar multiplet (A, ψ) and vector multiplet (A_μ, λ), put together, have the same field content as the on-shell $N = 2$ vector multiplet $(A, \psi, \lambda, A_\mu)$. The Lagrangian (8) contains all these fields but is not $N = 2$ supersymmetric. To make $(A, \psi, \lambda, A_\mu)$ form an $N = 2$ vector multiplet we need impose some restrictions on the $N = 1$ Lagrangian in (8). First, since A^a_μ and λ^a belong to the adjoint representation of the gauge group, A_i and ψ_i should also belong to the same representation as they are now in the same multiplet. Hence, $T^a_{ij} = -if^a_{ij}$ and the sets of indices $\{i\}$ and $\{a\}$ coincide. Second, since the two supersymmetry generators in the $N = 2$ algebra appear on the same footing, it should be the same case for the fermions ψ^a and λ^a in (8). So we should set the superpotential $\mathcal{W}$ to zero since it couples only to ψ^a. This condition also fixes the arbitrary relative normalization between the Yang-Mills part and the scalar part of the Lagrangian since it requires that the kinetic energy terms for both fermions have the same normalization. This is achieved by scaling $\Phi \to \Phi/g$ in (8). Conversely, if the Lagrangian (8) satisfies these conditions, then it is $N = 2$ supersymmetric. The terms containing the auxiliary fields are now

$$\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{2}DD + D\,[A^\dagger\,,\,A] + F^\dagger F\right)\,.$$

On eliminating D and F by using their equations of motion, we get the scalar potential

$$V = -\frac{1}{2g^2}\mathrm{Tr}\left([A^\dagger\,,\,A]^2\right)\,. \tag{9}$$

The full Lagrangian with $N = 2$ supersymmetry can now be written as

$$\begin{aligned}\mathcal{L} &= \frac{1}{8\pi}\mathrm{Im\,Tr}\left[\tau\left(\int d^2\theta\, W^\alpha W_\alpha + 2\int d^2\theta d^2\bar\theta\,\Phi^\dagger e^{-2V}\Phi\right)\right]\\ &= \frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g^2\frac{\theta}{32\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu} + (D_\mu A)^\dagger D^\mu A - \frac{1}{2}[A^\dagger\,,\,A]^2\\ &\qquad - i\,\lambda\sigma^\mu D_\mu\bar\lambda - i\,\bar\psi\bar\sigma^\mu D_\mu\psi - i\sqrt{2}\,[\lambda,\psi]\,A^\dagger - i\sqrt{2}\,[\bar\lambda,\bar\psi]\,A\Big)\,,\end{aligned} \tag{10}$$

The $N = 2$ Superspace Formulation: The $N = 2$ superspace is constructed by adding four more fermionic degrees of freedom, $\tilde\theta$ and $\bar{\tilde\theta}$, to the $N = 1$ superspace. A generic $N = 2$ superfield can be written as $F(x, \theta, \bar\theta, \tilde\theta, \bar{\tilde\theta})$. What we need is a superfield which has the same components as the $N = 2$ vector multiplet. We can make it by imposing chirality and reality constraints on a general $N = 2$ superfield. An $N = 2$ chiral superfield Ψ is defined by the constraints $\bar D_{\dot\alpha}\Psi = 0$ and $\bar{\tilde D}_{\dot\alpha}\Psi = 0$. Expand Ψ in powers of $\tilde\theta$:

$$\Psi = \Psi^{(1)}(\tilde y, \theta) + \sqrt{2}\tilde\theta^\alpha\Psi^{(2)}_\alpha(\tilde y, \theta) + \tilde\theta^\alpha\tilde\theta_\alpha\Psi^{(3)}(\tilde y, \theta)\,,$$

where, $\tilde y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta + i\tilde\theta\sigma^\mu\bar{\tilde\theta}$. The component $\Psi^{(1)}$ has the same form as the $N = 1$ chiral superfield Φ. The remaining two components are constrained by the reality condition. The result is that $\Psi^{(2)}_\alpha = W_\alpha(\tilde y, \theta)$ as given in (4), and

$$\Psi^{(3)}(\tilde y, \theta) = \Phi^\dagger(\tilde y - i\theta\sigma\bar\theta, \theta, \bar\theta) exp\left[2gV(\tilde y - i\theta\sigma\bar\theta, \theta, \bar\theta)\right]\,|_{\bar\theta\bar\theta}\ .$$

where $\Phi(\tilde{y} - i\theta\sigma\bar{\theta}, \theta, \bar{\theta})$ is in the form of (1). We can see that Ψ has the same field content as the $N = 2$ vector multiplet. In terms of the $N = 2$ superfield Ψ, the $N = 2$ Lagrangian (10) can be written as

$$\mathcal{L} = \frac{1}{4\pi} \mathrm{Im\,Tr} \int d^2\theta d^2\tilde{\theta}\, \frac{1}{2}\tau\Psi^2 \,. \tag{11}$$

The most general $N = 2$ Lagrangian for the gauge fields can be constructed by using the $N = 2$ chiral superfield Ψ (also called the $N = 2$ vector superfield): Corresponding to any holomorphic function $\mathcal{F}(\Psi)$, we can construct a Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{1}{4\pi} \mathrm{Im\,Tr} \int d^2\theta d^2\tilde{\theta}\, \mathcal{F}(\Psi) \\ &= \frac{1}{8\pi} \mathrm{Im} \left(\int d^2\theta\, \mathcal{F}_{ab}(\Phi) W^{a\alpha} W^b_\alpha + 2 \int d^2\theta d^2\bar{\theta}\, (\Phi^\dagger e^{2gV})^a \mathcal{F}_a(\Phi) \right) \,. \end{aligned} \tag{12}$$

where $\mathcal{F}_a(\Phi) = \partial\mathcal{F}/\partial\Phi^a$, $\mathcal{F}_{ab}(\Phi) = \partial^2\mathcal{F}/\partial\Phi^a\partial\Phi^b$ and $\mathcal{F}$ is referred to as the $N = 2$ prepotential. From the above, we can easily see that the Kähler potential is $\mathrm{Im}(\Phi^{\dagger a}\mathcal{F}_a(\Phi))$, which gives rise to a metric $g_{ab} = \mathrm{Im}(\partial_a\partial_b\mathcal{F})$ on the space of fields. If we demand renormalizability, then $\mathcal{F}$ has to be quadratic in Ψ as in (11). However, if we want to write a low-energy effective action, then renormalizability is not a criterion and $\mathcal{F}$ can have a more complicated form.

1.3 The $N = 2$ Supersymmetric Lagrangian with Matter Fields

Matter fields and gauge fields transform under different representations of the gauge group, so they cannot be part of the same multiplet. The $N = 2$ matter supermultiplet is called the hypermultiplet and contains one pair of complex scalars and one pair of two-component spinors, all transforming under the same representation of the gauge group. In $N = 1$ notation, a hypermultiplet contains a chiral superfield Q and an anti-chiral superfield $\widetilde{Q}^\dagger$, both transforming under the same representation N_c of the gauge group $SU(N_c)$. Denote the components of Q and $\widetilde{Q}$ by (q, ψ_q, F_q) and $(\tilde{q}, \psi_{\tilde{q}}, F_{\tilde{q}})$ respectively. The form of the Lagrangian for N_f hypermultiplets (labelled by an index i), interacting with a $N = 2$ vector multiplet, is given by

$$\begin{aligned} \mathcal{L} = \; & \int d\theta^4 \left(Q_i^\dagger e^{-2V} Q_i + \widetilde{Q}_i e^{2V} \widetilde{Q}_i^\dagger \right) + \int d\theta^2 \left(\sqrt{2}\widetilde{Q}_i \Phi Q_i + m_i \widetilde{Q}_i Q_i \right) + \, h.c. \\ & + \cdots \end{aligned} \tag{13}$$

where the dots represent the Lagrangian for the pure $N = 2$ vector multiplet.

Eliminating the auxiliary fields F_q and $F_{\tilde{q}}$ in the hypermultiplet results in a contribution to the scalar potential given by

$$V = \frac{1}{2} g^2 \sum_a D_a D^a \qquad \text{where } D^a = \sum_{i=1}^{N_f} \left(q_i^\dagger \lambda^a q_i - \tilde{q}_i \lambda^a \tilde{q}_i^\dagger \right) \,.$$

Here, λ^a are the gauge group generators in the fundamental representation.

1.4 Central Charges in the $N = 2$ Gauge Theory

In the supersymmetry algebra, central charge Z appears in the commutator of the supercharges Q^I_α which are space integrals of S^{I0}_α (Here, $S^{I\mu}_\alpha$ denotes the supercurrent). Thus, we first have to compute S^{I0}'s in terms of the basic fields, and then evaluate their commutators. A detailed calculation is made in [7], which gives the following results. For action (10), we have

$$Z = a\,(n_e + \tau_{cl} n_m)\,, \quad \text{where} \quad \tau_{\rm cl} = \frac{\theta}{2\pi} + \frac{4\pi {\rm i}}{{\rm g}^2}\,,$$

a is the value of A in the Higgs vacuum, and n_e, n_m are given by

$$g n_e = -\frac{1}{ag}\int d^3x \partial_i (F^{a0i} A^a)\,, \qquad \frac{4\pi}{g} n_m = -\frac{1}{ag}\int d^3x \partial_i (\widetilde{F}^{a0i} A^a)\,.$$

Using supersymmetry algebra this implies a mass bound

$$M \geq \sqrt{2}\,|Z| = \sqrt{2}\,|\,a\,(n_e + \tau_{cl} n_m)\,|$$

Note that when the θ-angle is set to zero, it is invariant under $n_e \leftrightarrow n_m$ accompanied by $\alpha \leftrightarrow 1/\alpha$ and $a \leftrightarrow a/\alpha$ where $\alpha = g^2/4\pi$. This provides some evidence for the Montonen-Olive conjecture of electromagnetic duality [2].

For the effective action (12) specified by a prepotential $\mathcal{F}$, the central charge takes the form

$$Z = a n_e + a_D n_m\,,$$

where $a_D = \partial\mathcal{F}/\partial a$.

When matter in the fundamental representation is added to the $N = 2$ pure gauge theory, we have

$$Z = n_e a + n_m a_D + \sum_{i=1}^{N_f} \frac{1}{\sqrt{2}}\, m_i S_i$$

where

$$S_i = \int d^3x \left(D_0 q_i^\dagger q_i + q_i D_0 q_i^\dagger - \frac{i}{2}\psi_{qi}^\dagger \psi_{qi} + \frac{i}{2}\psi_{qi}\psi_{qi}^\dagger - (q \to \tilde{q}, \psi_q \to \psi_{\tilde{q}}) \right)\,.$$

The index i here is not summed over.

2 The Seiberg-Witten Analysis of $N = 2$ Supersymmetric Pure Yang-Mills Theory

2.1 Parametrization of the Moduli Space

The scalar potential for the $N = 2$ supersymmetric Yang-Mills theory (10) is

$$V = \frac{1}{2g^2}{\rm Tr}\left([\phi^\dagger, \phi]^2\right)\,.$$

Classically, the Higgs vacuum is defined by $[\phi, \phi^\dagger] = 0$, which implies that ϕ takes values in the Cartan subalgebra of the gauge group: $\phi = \phi_i H^i$. Thus, generically, the gauge group G is broken to the subgroup H which is generated by elements from the Cartan subalgebra. Elements in G/H do not leave the Higgs vacuum invariant, but being gauge transformations, they relate physically equivalent vacua. On the other hand, once a given basis for the Cartan subalgebra is chosen, then different vacuum values of ϕ, within this subalgebra, correspond to different physical theories. Thus these degrees of freedom in ϕ parametrize the space of physically inequivalent vacua, or the moduli space of the theory. However, this parametrization is not good as there is still some residual gauge invariance: The coset G/H contains elements which, while not leaving the vacuum invariant, do not take ϕ out of the Cartan subalgebra. These transformations are precisely the Weyl reflections. Therefore, the correct parametrization of the moduli space is given not by ϕ, but by Weyl invariant functions constructed out of it.

The Weyl invariants are obtained from the characteristic equation,

$$\det(\lambda - \phi) = 0 \,.$$

Since Weyl reflections act on ϕ by conjugation, $\det(\lambda - \phi)$ is invariant. So if we expand this quantity as a polynomial of λ, then the coefficients are Weyl invariant quantities. For $SU(2)$, $\phi = \frac{1}{2} a \sigma_3$ and it can be easily checked that the desired Weyl invariant is $u = \mathrm{Tr}(\phi^2) = \frac{1}{2} a^2$.

In the above, we have treated ϕ classically. In quantum field theory, we parametrize the moduli space by the vacuum expectation values of the corresponding classical Weyl invariants. In our case, it is $u =< \mathrm{Tr}(\phi^2) >$, which at the classical limit, reduces to $a^2/2$.

2.2 R-Symmetry and its Breaking

The N-extended supersymmetry algebra is invariant under global $U(N)$ rotations of the N supercharges. Therefore, a supersymmetric theory should also exhibit such a global symmetry, called R-symmetry. The action of this global symmetry on the supercharges can be easily translated into a transformation of the superspace variables. For example, for $N = 1$, the $U(1)$ R-symmetry acts on the supercharge as $Q \to e^{-i\alpha} Q$. From this we can obtain its action on the superspace coordinates as $\theta \to e^{i\alpha}\theta$ and $\bar{\theta} \to e^{-i\alpha}\bar{\theta}$. For $N = 2$, we have a $U(1)$ which acts as above on the $N = 2$ superspace coordinates $\theta^I, \bar{\theta}_I$, along with an $SU(2)$ R-symmetry which rotates the index I of the supercharges. In order to keep the supersymmetric Lagrangian invariant under these transformations, we have to assign appropriate transformation properties to various superfields.

Action on the $N = 2$ Vector Multiplet: The $N = 2$ vector multiplet contains a vector field A_μ, two Weyl spinors λ, ψ and a scalar ϕ, all transforming in the adjoint representation of the gauge group. These components can be arranged as

$$\begin{array}{ccc} & A_\mu & \\ \lambda & & \psi \\ & \phi & \end{array} \,.$$

The $SU(2)_R$ transformation acts on the rows in the above diagram and rotates the fermions (λ, ψ) into each other while keeping A_μ and ϕ invariant. In the $N = 1$ formalism, this multiplet decomposes into a vector superfield $V(A_\mu, \lambda)$, and a chiral superfield $\Phi(\phi, \psi)$. Therefore, the only part of $SU(2)_R$ which remains manifest in the $N = 1$ language is a $U(1)_J$ subgroup which does not mix λ and ψ. This subgroup is generated by σ_3 and acts as $(\lambda, \psi) \to (e^{i\alpha}\lambda, e^{-i\alpha}\psi)$. The action of the $U(1)_{\mathcal{R}}$ and $U(1)_J$ on the $N = 1$ superfields are summarized below:

$$\begin{aligned} U(1)_{\mathcal{R}} \quad &: \Phi \to e^{2i\alpha}\Phi(e^{-i\alpha}\theta), \quad V \to V(e^{-i\alpha}\theta)\,, \\ U(1)_J \quad &: \Phi \to \Phi(e^{-i\alpha}\theta), \qquad V \to V(e^{-i\alpha}\theta)\,. \end{aligned} \tag{14}$$

Action on the $N = 2$ Hypermultiplet: In $N = 2$, the matter fields appear in hypermultiplets, each containing two complex scalars $(q, \tilde{q}^\dagger)$ and two Weyl fermions $(\psi_q, \tilde{\psi}_q^\dagger)$. All these components transform in the same (usually the fundamental) representation of the gauge group. The components of a hypermultiplet can be arranged as

$$\begin{array}{ccc} & \psi_q & \\ q & & \tilde{q}^\dagger \\ & \tilde{\psi}_q^\dagger & \end{array}.$$

Again, $SU(2)_R$ acts on the rows and thus rotates $q, \tilde{q}^\dagger$. In the $N = 1$ language, the hypermultiplet is decomposed in terms of chiral multiplets $Q(q, \psi_q)$, and $\tilde{Q}(\tilde{q}, \tilde{\psi}_q)$, which carry dual gauge quantum numbers. In this decomposition, only the $U(1)_J$ subgroup of $SU(2)_R$ is manifest. Moreover, in the $N = 1$ decomposition, a hypermultiplet interacting with a vector multiplet gives rise to a superpotential term

$$\mathcal{W} = \sqrt{2}\tilde{Q}\Phi Q.$$

Since $\mathcal{W}$ should carry two units of $U(1)_{\mathcal{R}}$ charge, Q and $\tilde{Q}$ are neutral. We summarize these transformations below:

$$\begin{aligned} U(1)_{\mathcal{R}}: \quad & Q \to Q(e^{-i\alpha}\theta)\,, \qquad \tilde{Q} \to \tilde{Q}(e^{-i\alpha}\theta)\,, \\ U(1)_J: \quad & Q \to e^{i\alpha}Q(e^{-i\alpha}\theta)\,, \quad \tilde{Q} \to e^{i\alpha}\tilde{Q}(e^{-i\alpha}\theta)\,. \end{aligned} \tag{15}$$

Note that we can combine the two-component spinors λ and $\bar{\psi}$ into a four-component Dirac spinor ψ_D. The spinor ψ_D transforms as $e^{i\alpha}\psi_D$ under $U(1)_J$ and as $e^{i\alpha\gamma_5}\psi_D$ under $U(1)_{\mathcal{R}}$. Similarly, $U(1)_{\mathcal{R}}$ acts as a chiral $U(1)$ on the Dirac spinor constructed out of ψ_q and $\tilde{\psi}_q^\dagger$, though now with the opposite charge. Thus $U(1)_{\mathcal{R}}$ is a chiral symmetry and is, therefore, broken by a chiral anomaly as will be discussed below.

Breaking of R-Symmetries: Classically, our theory has the full global $SU(2)_R \times U(1)_{\mathcal{R}}$ symmetry. But quantum mechanically, $U(1)_{\mathcal{R}}$ is broken to a discrete subgroup due to anomalies. To compute the anomaly for the gauge group $SU(N_c)$, note that by the index theorem, in the presence of an instanton, there is one zero-mode for each left moving fermion in the fundamental or antifundamental representation and $2N_c$ zero-modes for each left-handed fermion in the adjoint representation. A correlation function in this theory involves integrations over the fermionic collective coordinates corresponding to these zero-modes. For a correlator to be non-zero, it should contain enough fermion insertions to soak the zero-modes. Hence, in the presence of N_f flavors, the first non-vanishing correlator is

$$G = \langle \lambda(x_1)\cdots\lambda(x_{2N_c})\psi(y_1)\cdots\psi(y_{2N_c})\psi_q(z_1)\cdots\psi_q(z_{N_f})\widetilde{\psi}_q(u_1)\cdots\widetilde{\psi}_q(u_{N_f})\rangle \,. \tag{16}$$

Under $U(1)_{\mathcal{R}}$, G transforms as

$$G \rightarrow e^{i\alpha(4N_c-2N_f)}\, G\,.$$

Hence $U(1)_{\mathcal{R}}$ is broken to the discrete group $\mathbf{Z}_{4N_c-2N_f}$. For the pure $SU(2)$ Yang-Mills theory, $N_c = 2$, $N_f = 0$, so $U(1)_{\mathcal{R}} \rightarrow \mathbf{Z}_8$ and is represented by $e^{2\pi i\alpha}$, where $\alpha = n/8\,, n = 1, \cdots, 8$. Thus the global symmetry group is $SU(2)_R \times \mathbf{Z}_8$. However, note that the center of $SU(2)_R$, which acts as $(\lambda,\psi) \rightarrow e^{i\pi}(\lambda,\psi)$, is also contained in $\mathbf{Z}_8$ (corresponding to $n = 4$). Hence, the true symmetry group is

$$\left(SU(2)_R \times \mathbf{Z}_8\right)/\mathbf{Z}_2\,.$$

This surviving R-symmetry is broken further by the Higgs vacuum. The field ϕ^2 has charge 4 under $\mathbf{Z}_8$ and transforms to $e^{\pi i n}\phi^2$, which is invariant only for $n = 2, 4, 6, 8$. Therefore, if the vacuum is characterized by non-zero ϕ^2, then $\mathbf{Z}_8$ is broken down to $\mathbf{Z}_4$. All elements which do not keep ϕ^2 invariant, act as a $\mathbf{Z_2} : \phi^2 \rightarrow -\phi^2$. Therefore, the final R-symmetry group for the $SU(2)$ gauge theory is $(SU(2)_R \times \mathbf{Z_4})/\mathbf{Z_2}$.

2.3 Low-Energy Effective Action for $N = 2$ Gauge Theory

The $SU(2)$ gauge symmetry is spontaneously broken to $U(1)$ by a non-zero $\langle\phi\rangle$. At sufficiently low energies, non of the massive states will appear as physical states and an effective description of the theory can be given in terms of the massless fields alone by integrating out all the massive modes. In practice, this procedure is not easy to implement. Seiberg and Witten formulated an indirect procedure for determining the exact low-energy theory.

In terms of the massless chiral and vector superfields, the effective low-energy Lagrangian is

$$\mathcal{L}_{eff} = \frac{1}{4\pi}\,\mathrm{Im}\left[\int d^4\theta\,\frac{\partial\mathcal{F}(A)}{\partial A}\bar{A} + \int d^2\theta\,\frac{1}{2}\frac{\partial^2\mathcal{F}}{\partial A^2}W^\alpha W_\alpha\right]\,, \tag{17}$$

Thus the Kähler potential is $K = \mathrm{Im}\left(\bar{A}\partial\mathcal{F}(A)/\partial A\right)$. Denote the scalar component of the chiral superfield A by a, then the metric on the field space is

$$ds^2 = \mathrm{Im}(\tau)\, da d\bar{a}\,, \quad \text{where} \quad \tau(a) = \frac{\partial^2\mathcal{F}}{\partial a^2}. \tag{18}$$

Our aim in this subsection is to determine the perturbative form of $\mathcal{F}$ following [6].

Let's start from the exact microscopic theory (10) with gauge group $SU(2)$. This theory is asymptotically free and therefore at high energies one can perform reliable perturbative calculations. Note that the $U(1)_{\mathcal{R}}$ symmetry of this theory is broken by the chiral anomaly. Thus we have

$$\partial_\mu J_5^\mu = -\frac{N_c}{8\pi^2} F_{\mu\nu} \widetilde{F}^{\mu\nu} \, .$$

This implies that, to one-loop, under a $U(1)_{\mathcal{R}}$ transformation, the effective Lagrangian changes by

$$\delta \mathcal{L}_{eff} = -\frac{\alpha N_c}{8\pi^2} F \widetilde{F} \, . \tag{19}$$

Since $(32\pi^2)^{-1} \int F\widetilde{F}$ is an integer, the anomaly breaks $U(1)_{\mathcal{R}}$ to $\mathbf{Z}_{4N_c}$. The same result was obtained in the previous subsection from a consideration of fermion zero modes in an instanton background. The anomaly can be regarded as causing a shift in the θ-angle $\theta \to \theta + 4N_c\alpha$. In the following, we use this freedom to set θ to zero by an appropriate chiral rotation of the fermions.

The one-loop form of $\mathcal{F}$ can be determined from the requirement that under a $U(1)_{\mathcal{R}}$ transformation, the low-energy effective action (17) changes as in (19). The variation $\delta\mathcal{L}_{eff}$ could come only from terms in $\mathcal{L}_{eff}$ which are quadratic in $F_{\mu\nu}$. This gives

$$\frac{1}{16\pi} \mathrm{Im} \left[\mathcal{F}''(e^{2i\alpha}A)(-FF + iF\widetilde{F}) \right] = \frac{1}{16\pi} \mathrm{Im} \left[\mathcal{F}''(A)(-FF + iF\widetilde{F}) \right] - \frac{\alpha N_c}{8\pi^2} F\widetilde{F} \, .$$

The form of the prepotential is therefore restricted by

$$\mathcal{F}''(e^{2i\alpha}A) = \mathcal{F}''(A) - \frac{2\alpha N_c}{\pi} \, ,$$

or, for infinitesimal α, by

$$\frac{\partial^3 \mathcal{F}}{\partial A^3} = \frac{N_c}{\pi} \frac{i}{A} \, .$$

Integrating the above equation gives the one-loop expression for the prepotential

$$\mathcal{F}_{1-loop}(A) = \frac{i}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2} \, . \tag{20}$$

Here, Λ is a fixed dynamically generated scale. It is known that, due to $N = 2$ supersymmetry, the above one-loop expression for the prepotential does not receive higher order perturbative corrections. This is related to the fact that in this theory the perturbative β-function is only a one-loop effect.

Although the prepotential given in (20) is exact in perturbation theory, it does receive non-perturbative corrections due to instanton effects as argued in [6]. The correction to $\mathcal{F}$ coming from a configuration of instanton number k is proportional to the k-instanton factor $exp(-8\pi^2 k/g^2)$ (since the prepotential is a holomorphic function, it cannot receive corrections from anti-instanton configurations). Using the one-loop β-function of the theory ($N_c = 2$) given by $\beta(g) = -g^3/4\pi^2$, the k-instanton factor can be written as

$$e^{-8\pi^2 k/g^2} = \left(\frac{\Lambda}{a}\right)^{4k} \, .$$

Following the approach of Seiberg in [9, 10], we note that the broken $U(1)_{\mathcal{R}}$ symmetry is restored if we assign a charge of 2 to Λ. With this modification, the prepotential should transform under $U(1)_{\mathcal{R}}$ as a field of charge 4, without a non-homogeneous term. This implies that the k-instanton correction should also be proportional to A^2. Putting these together, the prepotential including generic non-perturbative corrections can be written as

$$\mathcal{F} = \frac{i}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{A}\right)^{4k} A^2 \,.$$

The coefficients $\mathcal{F}_k$ are not field dependent due to the fact that in a supersymmetric theory, instantons contribute to the path integral only through zero-modes [8].

The one-loop expression for $\mathcal{F}$, coupled with the fact that a well defined theory should have a positive kinetic energy term, leads to very interesting consequences. For large $|a|$, using (20), we can calculate $\tau(a) = \frac{i}{\pi}(\ln \frac{a^2}{\Lambda^2} + 3)$. This is a multi-valued function, though the metric on the field space given by $\mathrm{Im}\tau$ is single valued. However, since $\mathrm{Im}\tau(a)$ is a harmonic function, it cannot have a global minimum. Thus, if it is globally defined, it cannot be positive everywhere. Therefore, the positivity of the kinetic energy requires that $\tau(a)$ is defined only locally. This means that in the region of the complex plane where $\tau(a)$ becomes negative, one needs a different description of the theory.

2.4 Duality

To find the duality transformations which relate different descriptions of the same theory, we consider the gauge field terms in the bosonic part of the action. Working in the Minkowski space with conventions $(F_{\mu\nu})^2 = -(\widetilde{F}_{\mu\nu})^2$ and $\widetilde{\widetilde{F}} = -F$, these terms can be written as

$$\frac{1}{32\pi} \mathrm{Im} \int \tau(a)(F + i\widetilde{F})^2 = \frac{1}{16\pi} \mathrm{Im} \int \tau(a)(F^2 + i\widetilde{F}F) \,.$$

Now we regard F as an independent field and implement the Bianchi identity $dF = 0$ by introducing a Lagrange multiplier vector field V_D. A magnetic monopole satisfies $\epsilon^{0\mu\nu\rho}\partial_\mu F_{\nu\rho} = 8\pi\delta^{(3)}(x)$. The Lagrange multiplier term can be constructed by coupling V_D to a monopole:

$$\frac{1}{8\pi} \int V_{D\mu} \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = \frac{1}{8\pi} \int \widetilde{F}_D F = \frac{1}{16\pi} \mathrm{Re} \int (\widetilde{F}_D - iF_D)(F + i\widetilde{F}) \,,$$

where, $F_{D\mu\nu} = \partial_\mu V_{D\nu} - \partial_\nu V_{D\mu}$. Adding this to the gauge field action and integrating over F gives the dual theory

$$\frac{1}{32\pi} \mathrm{Im} \int \left(-\frac{1}{\tau}\right) (F_D + i\widetilde{F}_D)^2 = \frac{1}{16\pi} \mathrm{Im} \int \left(-\frac{1}{\tau}\right) (F_D^2 + i\widetilde{F}_D F_D) \,.$$

The dualization can also be performed in an $N = 1$ supersymmetric language. In this case, the relevant term in the action is

$$\frac{1}{8\pi} \mathrm{Im} \int d^2\theta \tau(A) W^2 \,.$$

The Bianchi identity is replaced by $\mathrm{Im}\mathcal{D}W = 0$. This can be implemented by introducing a vector superfield V_D and the corresponding Lagrange multiplier term becomes

$$\frac{1}{4\pi}\,\mathrm{Im}\int d^4x d^4\theta\, V_D \mathcal{D} W = \frac{1}{4\pi}\,\mathrm{Re}\int d^4x d^4\theta\, i\mathcal{D}V_D W = -\frac{1}{4\pi}\,\mathrm{Im}\int d^4x d^2\theta\, W_D W\,.$$

Adding this to the action and integrating out W, gives the dual action

$$\frac{1}{8\pi}\,\mathrm{Im}\int d^2\theta \left(-\frac{1}{\tau(A)} W_D^2\right)\,.$$

Thus, the effect of the duality transformation is to replace a gauge field which couples to electric charges by a dual gauge field which couples to magnetic charges, and at the same time, transform the gauge coupling as

$$\tau \to \tau_D = -\frac{1}{\tau}\,. \tag{21}$$

We recognize this as the electric-magnetic duality. Also, note that the action is invariant under the replacement $\tau \to \tau + 1$. This transformation, along with the one in (21), generates the $SL(2,\mathrm{Z})$ group which, therefore, is the full duality group of our theory. This group acts on τ as

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \tag{22}$$

where, $ad - bc = 1$ and $a, b, c, d \in \mathrm{Z}$.

$N = 2$ supersymmetry relates the dual description of the gauge fields to a dual description for the scalar fields. To see this, define $h(A) = \partial\mathcal{F}/\partial A$, then $\tau(A) = \partial h(A)/\partial A$ and the scalar kinetic energy term becomes $\mathrm{Im}\int d^4\theta\, h(A)\bar{A}$. For the dual theory corresponding to (21), let's introduce the corresponding dual variables $A_D, \mathcal{F}_D, h_D(A_D)$ and τ_D. Then, equation (21) implies that $A_D = h = \partial\mathcal{F}/\partial A$ and $h_D = -A$. Under this transformation, the scalar kinetic energy term retains its form,

$$\mathrm{Im}\int d^4\theta\, h(A)\bar{A} = \mathrm{Im}\int d^4\theta\, h_D(A_D)\bar{A}_D\,.$$

We now consider the effect of the full $SL(2,Z)$ group on A and $\mathcal{F}$, or equivalently on A and $A_D = \partial\mathcal{F}/\partial A$. The transformation (22) implies that

$$\begin{pmatrix} A_D \\ A \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A_D \\ A \end{pmatrix}\,. \tag{23}$$

In general, we could also add a constant column matrix C to the right-hand side of the above equation. However, for non-zero C, the BPS mass formula for the theory in the absence of matter is not invariant under the above transformation. Thus in this case we should set $C = 0$. (In the presence of matter fields, the BPS mass formula is modified and a non-zero C is allowed.) On the space of the scalar fields, the transformations $\tau \to -1/\tau$ and $\tau \to \tau + 1$ are implemented by the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{24}$$

These matrices generate the $SL(2, \mathrm{Z})$ group.

Let us now come back to the metric on the moduli space as given by (18). In terms of a_D,

$$ds^2 = \mathrm{Im}\, da_D d\bar{a} = -\frac{i}{2}(da_D d\bar{a} - da d\bar{a}_D)\,,$$

which is invariant under the $SL(2, \mathrm{Z})$ transformations. The moduli space $\mathcal{M}$ is a complex one dimensional manifold with a holomorphic coordinate u. Let a and a_D be the coordinates on a space $X \cong \mathrm{C}^2$ on which we can choose a symplectic form $\omega = \mathrm{Im}\, da_D \wedge d\bar{a}$. The functions $(a_D(u), a(u))$ give a map f from $\mathcal{M}$ to X. In other words, they determine a section of X regarded as an $SL(2, \mathrm{Z})$ bundle over $\mathcal{M}$. In terms of u, the metric on the moduli space is

$$ds^2 = \mathrm{Im}\, \frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}}\, du d\bar{u} = -\frac{i}{2}\left(\frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}} - \frac{d\bar{a}_D}{d\bar{u}}\frac{da}{du}\right) du d\bar{u}\,.$$

Choosing $u = a$, we get back to the original formula. $(a_D(u), a(u))$ cannot be arbitrary since the metric should be positive.

2.5 Dyon Masses

Magnetic Monopole is a static and finite energy semi-classical solution to the ordinary Yang-Mills-Higgs theory [11]. Because of the θ term, the magnetic charge will also contribute to the electric charge, so we have dyons. For the microspcopic $SU(2)$ theory, the BPS bound is given by $M \geq \sqrt{2}|Z|$, where,

$$Z = a(n_e + \tau_{cl}\, n_m)\,,$$

and $\tau_{cl} = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g^2}$. All states that saturate this bound fall in a short multiplet of the $N = 2$ algebra. In a more general $N = 2$ theory, this formula is slightly modified. Suppose, the theory contains matter fields in hypermultiplets. When $a \neq 0$, these fields, which in the $N = 1$ formalism are described by chiral fields $M, \widetilde{M}$, become massive. If a hypermultiplet carries charge n_e, then its coupling to the chiral field A is uniquely fixed by $N = 2$ supersymmetry as

$$\sqrt{2}\, n_e\, AM\widetilde{M}\,.$$

From this we can easily see that for such a state, $Z = a\, n_e$. On the other hand, if we consider a magnetic monopole carrying n_m units of magnetic charge, then a similar manipulation leads to the BPS bound $\sqrt{2}|n_m\, a_D|$ or, equivalently, $Z = n_m\, a_D$. In general, for dyons

$$Z = an_e + a_D n_m\,. \tag{25}$$

Since masses are physically observable, the mass formula should be invariant under the monodromies on the moduli space. Therefore, if $v = (a_D, a)^T$ transforms to Mv, then the vector $w = (n_m, n_e)$ should transform to wM^{-1}. Note that this requires M to be an integral matrix.

2.6 $U(1)$ β-function

If we have Weyl fermions with charge Q_f and complex scalars with charge Q_s, then their contribution to the $U(1)$ β-function is

$$\beta(g) \equiv \mu \frac{d}{d\mu} g = \frac{g^3}{16\pi^2} \left(\sum_f \frac{2}{3} Q_f^2 + \sum_s \frac{1}{3} Q_s^2 \right) .$$

Denote the coefficient of g^3 by b and define $\alpha = g^2/4\pi$, then the above equation can be rewritten as

$$\mu \frac{d}{d\mu} \left(\frac{1}{\alpha} \right) = -8\pi b \, .$$

Consider hypermultiplet which is a reduced multiplet of $N = 2$ with spin $\leq 1/2$. In $N = 1$ language, this is described by chiral superfields $M, \widetilde{M}$ and contains two Weyl fermions and two complex scalars, all with the same charge Q. Hence we get

$$b = \frac{1}{16\pi^2} \, Q^2 \left(2 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3} \right) = \frac{1}{8\pi^2} \, Q^2 \, .$$

Now remember that, using the anomaly, we have set the θ-parameter to zero by a chiral rotation of the fermions. As a result we have $\tau = i/\alpha$, so that,

$$\mu \frac{d\tau}{d\mu} = -\frac{i}{\pi} Q^2 \, .$$

Identifying μ with the natural scale of the theory which is a and setting $Q = 1$, we obtain

$$\tau \simeq -\frac{i}{\pi} \ln \frac{a}{\Lambda} \, .$$

If we are interested in the contribution from a monopole multiplet, then we can perform the above calculation in terms of the dual variables. The answer then becomes $\tau_D \simeq -(i/\pi) \ln{(a_D/\Lambda)}$.

2.7 Monodromies on the Moduli Space of the $SU(2)$ Theory

Monodromy at large u: At large $|a|$, the theory is asymptotically free and $u = \frac{1}{2}a^2$. To a good approximation, the prepotential is given by the one-loop formula $\mathcal{F}(a) = (i/2\pi)a^2 \ln{(a^2/\Lambda^2)}$, so

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = \frac{2ia}{\pi} \ln{(\frac{a}{\Lambda})} + \frac{ia}{\pi} \, .$$

If we make a close loop on the u-plane around $u = 0$, we get $\ln u \rightarrow \ln u + 2\pi i$, and $\ln a \rightarrow \ln a + i\pi$, hence,

$$\begin{aligned} a_D &\rightarrow -a_D + 2a \, , \\ a &\rightarrow -a \, . \end{aligned}$$

This is implemented by the monodromy matrix

$$M_\infty \;=\; PT^{-2} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \tag{26}$$

acting on $(a_D, a)^T$. Here, P is the negative of the identity element in $SL(2,\mathrm{Z})$ and T is as defined in (24). Under the action of this monodromy, the magnetic and electric quantum numbers of BPS states change as $(n_m, n_e) \to (-n_m, -n_e - 2n_m)$ so that the mass formula is unchanged.

Monodromies at finite u: The monodromy at $u = \infty$ implies that there exist other monodromies somewhere else on the u-plane. If these monodromies commute with M_∞, then a^2 is a good global coordinate. But then the positivity of the kinetic energy is violated. To get a non-Abelian monodromy group, we need at least two singularities on the u-plane, and at finite u, with non-trivial monodromies around them. These singularities will be related by the broken discrete symmetry $u \to -u$. We make this minimal assumption on the number of extra singularities. Then a loop enclosing both these singularities should reproduce the monodromy M_∞.

Let's discuss the origin of these singularities on the moduli space. To obtain the Wilsonian low-energy effective action, we have integrated out all massive states in the theory. The massive particle loops then give rise to a non-trivial metric on the moduli space. But the values of the masses depend on u, and it may so happen that for certain values of u, some of the masses become zero. Then, at these points, we end up integrating out massless states and thus create singularities on the moduli space. The nature of a singularity, *i.e.,* the monodromy associated with it, depends on the properties of the particle which becomes massless at the singularity. But for finite u, the mass spectrum as a function of u is unknown, so the singularities cannot be found in a straightforward way. The way to proceed is to assume that some generic states become massless at certain values of u (say $u = 1$ and $u = -1$) and find the corresponding monodromies (M_1 and M_{-1}). The massless states are then specified by the condition $M_1 M_{-1} = M_\infty$.

One may expect that the massive gauge boson multiplets contribute to the singularity when they become massless. But as argued in [4], a spin-1 multiplet becoming massless does not lead to a consistent picture. We will not repeat this argument here.

The only other massive states in the theory with spin $\leq 1/2$ are monopoles and dyons, which, due to the spin condition, belong to short $N = 2$ multiplets and, therefore, are BPS states. These states are described by hypermultiplets. Seiberg and Witten suggested that the singularities arise when some of these non-perturbative states become massless. The problem now is to calculate the associated monodromies. Note that the hypermultiplets for monopoles and dyons cannot be coupled to the fundamental fields of our theory in a local way. However, in the subsection on duality it was seen that it is possible to go to a dual description of the theory in which some dual gauge fields couple to the monopoles or dyons in exactly the same way that the usual gauge fields couple to a particle of unit electric charge. Thus we only have to calculate the monodromy when a massive electrically charged hypermultiplet becomes massless and then, using the duality transformation, find the monodromy for a generic monopole or dyon. Let us consider a dual description of the theory in which a certain monopole or dyon appears as an elementary state, and label this description by the letter q. Near the point where this state is massless, all massive fields can be integrated out and the theory is essentially a $U(1)$ theory coupled to a hypermultiplet. Denote the *VEV* of the scalar field in this description of the theory by $a(q)$, then the mass of a BPS state of unit electric charge goes to zero when $a(q) = 0$ at some $u = u_q$. Thus, near this point, $a(q)$ is a good local coordinate and can be expanded as $a(q) \approx c_q\,(u-u_q)$. Moreover, near this point, the one-loop $U(1)$ β-function implies:

$$\tau(a(q)) = -\frac{i}{\pi}\ln\frac{a(q)}{\Lambda}\,,$$

from which we obtain

$$a_D(q) = -\frac{i}{\pi}\,a(q)\ln\frac{a(q)}{\Lambda} + \frac{i}{\pi}\,.$$

Moving on a closed loop around u_q so that $(u-u_q) \to e^{2\pi i}(u-u_q)$, we get the monodromy

$$\begin{array}{rcl} a_D(q) & \to & a_D(q) + 2a(q)\,, \\ a(q) & \to & a(q)\,. \end{array} \qquad (27)$$

Let's now calculate the monodromy when a (n_m, n_e) dyon becomes massless (the dyon is stable or marginally stable if n_m and n_e are coprime). The first step is to find a dual description of the theory in which this dyon appears as an elementary state of charge $(0,1)$. Under a generic $SL(2,\mathrm{Z})$ transformation we get a $(n_m(q), n_e(q))$ dyon with

$$\left(\begin{array}{c} a_D(q) \\ a(q) \end{array}\right) = \left(\begin{array}{c} \alpha a_D + \beta a \\ \gamma a_D + \delta a \end{array}\right), \qquad \left(\begin{array}{c} n_m(q) \\ n_e(q) \end{array}\right) = \left(\begin{array}{c} n_m\delta - n_e\gamma \\ -n_m\beta + n_e\alpha \end{array}\right), \qquad (28)$$

where $\alpha\delta - \beta\gamma = 1$, so that $Z = n_m a_D + n_e a$ is invariant. Choose the parameters such that $n_m(q) = 0, n_e(q) = 1$. With this choice, $(a_D(q), a(q))$ become the variables in terms of which the dyon couples to the $SL(2,\mathrm{Z})$ transformed gauge field in the same way that the unit electric charge couples to usual gauge fields. In particular, when the dyon becomes massless at some point on the moduli space, the associated monodromy is given by (27). Inverting the first equation in (28), we get $a_D = -\beta a(q) + n_e a_D(q)$ and $a = \alpha a(q) - n_m a_D(q)$. Thus we can easily find the action of the monodromy on the original variables as

$$\left(\begin{array}{c} a_D \\ a \end{array}\right) \to \left(\begin{array}{cc} 1 + 2n_e n_m & 2n_e^2 \\ -2n_m^2 & 1 - 2n_e n_m \end{array}\right)\left(\begin{array}{c} a_D \\ a \end{array}\right). \qquad (29)$$

Denote the monodromy matrix by $M(n_m, n_e)$. (Note that $\mathrm{Tr}\,M(n_m, n_e) = 2$, so the monodromy always belongs to the parabolic subgroup of $SL(2,\mathrm{Z})$.)

Now let's calculate the monodromies at $u = \pm 1$. Suppose an (m,n) dyon becomes massless at $u = 1$ and a (m', n') dyon becomes massless at $u = -1$. Then the associated monodromies should satisfy

$$M_1(m,n)\,M_{-1}(m',n') = M_\infty \qquad (30)$$

Using (26) and (29), this can be written as

$$\left(\begin{array}{cc} 1 + 2mn & 2n^2 \\ -2m^2 & 1 - 2mn \end{array}\right) = \left(\begin{array}{cc} -1 & 2 \\ 0 & -1 \end{array}\right)\left(\begin{array}{cc} 1 - 2m'n' & -2n'^2 \\ 2m'^2 & 1 + 2m'n' \end{array}\right),$$

leading to the equations

$$\begin{aligned} 1+mn &= m'n' + 2m'^2 \,, \\ m^2 &= m'^2 \,, \\ n^2 &= n'^2 + 1 + 2m'n' \,, \\ 1-mn &= -m'n' \,. \end{aligned}$$

These imply that $m = \pm 1$ and $m' = \pm 1$. For each combination of (m, m'), we can easily determine n' in terms of n and get the following possible sets of solutions

$$\begin{array}{lccccccc} (m,n): & (1,n) & , & (-1,n) & , & (-1,n) & , & (1,n)\,, \\ (m',n'): & (1,n-1) & , & (1,-n-1) & , & (-1,n+1) & , & (-1,-n+1)\,. \end{array}$$

There do not seem to be any solutions where M_∞ is factorized into a product of more than two such parabolic $M(m,n)$. Note that the solution allows only dyons of unit magnetic charge to contribute to the monodromy. This is consistent with the result that, semiclassically, only these dyons are stable.

In general, under the action of the monodromy, the quantum numbers of dyons will change. However, we expect that the dyon which becomes massless and is the source of the singularity should remain invariant under the monodromy. The eigenvalue equation

$$(q_m, q_e) \begin{pmatrix} 1+2mn & 2n^2 \\ -2m^2 & 1-2mn \end{pmatrix} = (q_m, q_e)\,, \tag{31}$$

leads to $nq_m - mq_e = 0$, if m and n do not vanish simultaneously. Clearly, $q_m = m, q_n = n$ is a solution of this equation. If we restrict ourselves to stable dyons, then this is the only solution. Thus, knowing the monodromy, we can find the dyon which gives rise to it.

The simplest solution to the equation (30) corresponds to $m = m' = 1, n = 0, n' = -1$. The monodromy matrices are then given by

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}. \tag{32}$$

A comparison with (31) implies that M_1 arises due to a monopole becoming massless at $u = 1$ and M_{-1} arises when a $(1, -1)$ dyon becomes massless at $u = -1$. At the point where the monopoles become massless, we have $a_D = 0$ and where the $(1, -1)$ dyon becomes massless, we have $a - a_D = 0$ as is evident from $Z = n_m a_D + n_e a$. Note that the monodromy at infinity, M_∞, shifts the electric charge by 2 units. Hence, at the points where condensation takes place, the electric charge is really defined modulo 2. We can conjugate the representation of the fundamental group by M_∞^n.

2.8 The Solution of the Model

The moduli space $\mathcal{M}$ is the u-plane with singularities at $1, -1, \infty$ and a $\mathbf{Z}_2$ symmetry acting as $u \to -u$ (Fig. 3). Over this punctured plane there is a flat $SL(2, \mathbf{Z})$ bundle V, which has

$(a_D, a)^T$ as a section. Around the singularities this bundle has monodromies

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \tag{33}$$

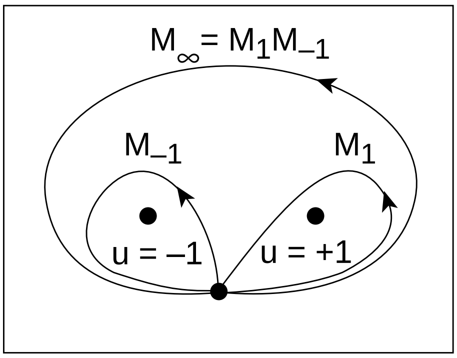

Figure 1

The quantities $(a_D(u), a(u))$ have the following asymptotic behaviour

$$u \approx \infty : \quad \begin{cases} a \cong \sqrt{2u} \\ a_D \approx i \frac{\sqrt{2u}}{\pi} \ln u \end{cases},$$

$$u = 1 : \quad \begin{cases} a_D \approx c_0(u-1) \\ a \approx a_0 + \frac{i}{\pi} a_D \ln a_D \end{cases}.$$

where, a_0 and c_0 are constants. For $u = -1$ we get a behavior similar to $u = 1$ but with a_D replaced by $a - a_D$. The metric on the moduli space is $ds^2 = \mathrm{Im}(\tau)\,|da|^2$ with

$$\tau = \frac{da_D/du}{da/du}. \tag{34}$$

To insure positivity of kinetic energy, $\mathrm{Im}(\tau)$ should be positive definite. The monodromies generate a subgroup $\Gamma(2)$ of $SL(2,\mathrm{Z})$ and, in fact, the u-plane with its singularities is the quotient of the upper half plane H by $\Gamma(2)$. This quotient has three cusps corresponding to the three singularities.

The space $H/\Gamma(2)$, or the u-plane, parametrizes the family of curves E_u described by the equation

$$y^2 = (x-1)(x+1)(x-u), \tag{35}$$

where x is a complex variable. Note that this equation is invariant under the transformations

$$w : \{u \to -u\,, x \to -x\,, y \to \pm iy\},$$

that generate a $\mathbf{Z}_4$ symmetry, out of which only a $\mathbf{Z}_2$ subgroup acts on u. This is the same as the symmetry structure on $\mathcal{M}$. The curve basically is the x-space with topology determined by the requirement that y is a single valued function. Since the equation is quadratic in y, if we move along a close loop on the x-space around anyone of the three zeros of y, then we get $y \to -y$. The same is also true for a loop which contains all the zeros (because there is an odd number of zeros), or equivalently, a loop around the point $x = \infty$. Therefore, if y is to be a single valued function, then the x-space should be a double cover of the extended complex plane $\mathbf{C}$ with four branch points at $x = -1, 1, u, \infty$ which are joined pairwise by two cuts. To fix attention, consider one branch cut from -1 to 1, and another from u to ∞. The two sheets are joined along these cuts so that on crossing a cut, we move from one sheet to the other. It is on this space that y is single valued.

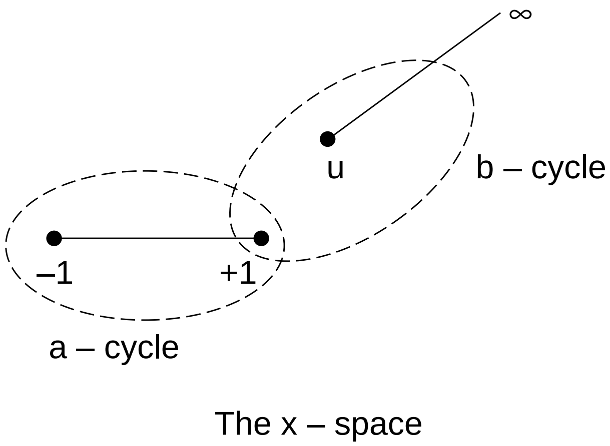

Figure 2

The x-space so obtained is nothing but a genus 1 Riemann surface. This can be easily visualized as follows: On a torus, draw a circle c_1 along the a-cycle and translate this along the torus to get a circle c_2 (Fig. 5(a)). Now, squash the circles c_1 and c_2 into line segments l_1 and l_2 (Fig. 5(b)). This divides the torus into two halves joined along these segments. If we now cut open both the two halves, the surface we get is the same as the x-space described above with l_1 and l_2 as the two branch cuts and with the point at infinity mapped to a point at finite x (Fig. 5(c)).

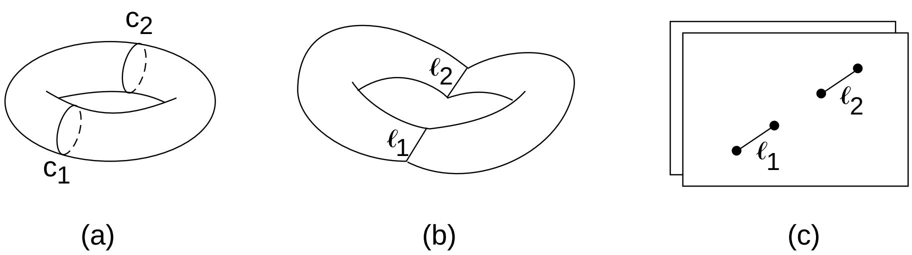

Figure 3

From this, it is clear that a loop on the x-plane that goes around one of the two cuts corresponds to the a-cycle of the torus and a loop which intersects both cuts corresponds to the b-cycle on the torus. Note that when two of the branch points on the x-space coincide, a cycle on the curve E_u shrinks to zero size and the curve becomes singular. For example, in our case, if $u = \infty$, the a-cycle shrinks to zero size and when $u = 1$, this happens to the b-cycle. Thus the singularities on the u-plane are at the points where a curve in the family E_u develops a vanishing cycle.

To identify a_D and a, on the genus 1 Riemann surface E_u, we pick up two independent one cycles γ_1 and γ_2, normalized such that their intersection number is one. These cycles, which continuously vary with u, form a local basis for the first homology group $V_u = H^1(E_u, \mathrm{C})$ of E_u. A cycle can be paired with elements λ from the first cohomology group

$$\gamma \to \oint_\gamma \lambda\,.$$

λ can be thought of as a meromorphic $(1,0)$-form on E_u with vanishing residue, modulo exact forms. The vanishing of the residue insures that the pairing is invariant under continuous deformations of γ even across poles of λ. By virtue of this pairing, we can also regard λ as an element of V_u. For the one-forms on E_u, we can choose a basis

$$\lambda_1 = \frac{dx}{y}\,, \qquad \lambda_2 = \frac{xdx}{y}\,.$$

Up to scalar multiplication, λ_1 is the unique holomorphic differential on E_u and if we define

$$b_i = \oint_{\gamma_i} \lambda_1\,, \qquad \text{for} \quad i = 1, 2\,,$$

then the torus is characterized by a parameter

$$\tau_u = b_1/b_2\,, \qquad \text{with} \quad \mathrm{Im}(\tau_u) > 0\,.$$

Let us consider an arbitrary section

$$\lambda = a_1(u)\lambda_1 + a_2(u)\lambda_2\,,$$

of V_u and, for the moment, make the identification

$$a_D = \oint_{\gamma_1} \lambda\,, \qquad a = \oint_{\gamma_2} \lambda\,.$$

If λ is a form with vanishing residue then, on circling a singularity, a_D and a transform in the right way, simply according to how γ_1 and γ_2 transform under a subgroup of $SL(2,Z)$. If λ has a pole with a non-vanishing residue, then it is possible that the integration path may move across this pole and, as a result, a_D and a may no longer transform under a pure $SL(2,Z)$. This second possibility is of course not consistent with the symmetries of the BPS mass formula in the absence of fermions with non-zero bare masses. Therefore, λ should not have a pole with a non-vanishing residue (In the presence of fermions with non-zero bare masses, the situation is different). The above identification of a_D and a implies that

$$\frac{da_D}{du} = \oint_{\gamma_1} \frac{d\lambda}{du}\,, \qquad \frac{da}{du} = \oint_{\gamma_2} \frac{d\lambda}{du}\,.$$

To fix the arbitrariness in λ, we use the condition $\mathrm{Im}\tau > 0$ for the metric on $\mathcal{M}$ as defined in (34). First, suppose that

$$\frac{d\lambda}{du} = f(u)\lambda_1 = f(u)\frac{dx}{y}\,.$$

Then,

$$\frac{da_D}{du} = f(u)\, b_1\,, \qquad \frac{da}{du} = f(u)\, b_2\,,$$

so that

$$\tau = \frac{b_1}{b_2} = \tau_u\,.$$

Since $\mathrm{Im}\,\tau_u > 0$, we get $\mathrm{Im}\,\tau > 0$. As argued by Seiberg and Witten, the converse is also true, so $d\lambda/du$ does not depend on λ_2. The function $f(u)$ is fixed by the asymptotic behavior of the theory near the singularities on the u plane and is given by $f(u) = -\sqrt{2}/4\pi$. With this, we can obtain λ as

$$\lambda = \frac{\sqrt{2}}{2\pi}\frac{dx\sqrt{x-u}}{\sqrt{x^2-1}} = \frac{\sqrt{2}}{2\pi}\frac{dxy}{x^2-1} = \frac{\sqrt{2}}{2\pi}\frac{dx}{y}(x-u)\,.$$

To calculate a and a_D, we have to choose a specific basis of one-cycles on E_u. We identify γ_2 with the a-cycle on the torus, or equivalently, with a curve which loops around the points -1 and 1 on the x plane. We can deform this curve so that it lies entirely along the cut from -1 to 1 and back. Thus, $a(u)$ is given by

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{dx\sqrt{x-u}}{\sqrt{x^2-1}}\,. \tag{36}$$

For γ_1, we choose the curve which loops around the points 1 and u and get

$$a_D(u) = \frac{\sqrt{2}}{\pi}\int_{1}^{u}\frac{dx\sqrt{x-u}}{\sqrt{x^2-1}}\,. \tag{37}$$

It can be checked that with this choice of the one-cycles, a and a_D have the desired behavior near the singularities [4].

References

[1] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, 2nd Edition (1992).

[2] C. Montonen and D. Olive, Phys. Lett. **72B** (1977) 117.

[3] A. Bilal, *Duality in N=2 SUSY SU(2) Yang-Mills Theory: A Pedagogical Introduction to the Work of Seiberg and Witten*, hep-th/9601007.

[4] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19.

[5] N. Seiberg and E. Witten, Nucl. Phys. **B431** (1994) 484.

[6] N. Seiberg, Phys. Lett. **206B** (1988) 75.

[7] L. Alvarez-Gaume and S. Hassan, hep-th/9701069.

[8] A. D'adda and P. DiVecchia, Phys. Lett. **73B** (1978) 162.

[9] N. Seiberg, Phys. Lett. **318B** (1993) 469.

[10] N. Seiberg, *Power of Holomorphy - Exact Results in 4-D Supersymmetric Gauge Theories*, hep-th/9408013.

[11] J. Harvey, *Magnetic Monopoles, Duality, and Supersymmetry*, hep-th/9603086