Problem Set 1

Physics 487

Due January 26

Some abbreviations: WB - Wess and Bagger

1. Consider the action for a single particle discussed in lecture (modulo factors of 2):

$$S = \int dt \left\{ \frac{1}{2}\dot{x}^2 + i\psi^*\dot{\psi} - \frac{1}{2}(W')^2 - W''\psi^*\psi \right\}.$$

In order that the bosonic potential go to infinity as $|x| \to \infty$, we demand that the superpotential $W \to \pm\infty$ as $|x| \to \infty$. Compute the supersymmetric index as a function of the possible asymptotic behaviors of W. Show that your answer agrees with a direct analysis of the Schrödinger equation for the cases $W = x^3$ and $W = x^4$.

2. To get familiar with the notation used by WB, verify the following relations for general anticommuting variables:

$$\begin{aligned}
(\sigma^m\bar{\sigma}^n + \sigma^n\bar{\sigma}^m)_\alpha^{\ \beta} &= -2\eta^{mn}\delta_\alpha^{\ \beta} \\
(\bar{\sigma}^m\sigma^n + \bar{\sigma}^n\sigma^m)^{\dot{\alpha}}_{\ \dot{\beta}} &= -2\eta^{mn}\delta^{\dot{\alpha}}_{\ \dot{\beta}} \\
\mathrm{Tr}(\sigma^m\bar{\sigma}^n) &= -2\eta^{mn} \\
\theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta \\
\theta_\alpha\theta_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta \\
\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \\
\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} &= -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \\
(\theta\phi)(\theta\psi) &= \frac{1}{2}(\phi\psi)(\theta\theta) \\
(\psi\sigma^m\bar{\chi})(\psi\sigma_m\bar{\rho}) &= -2(\psi\phi)(\bar{\chi}\bar{\rho}) \\
\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta(\theta\theta) &= 4
\end{aligned} \tag{1}$$

3. WB p.20 problem 2.

4. Show that if a superfield satisfies both $D_\alpha\Phi = 0$ and $\bar{D}_{\dot{\alpha}}\Phi = 0$ then it is a constant, independent of x, θ, $\bar{\theta}$.

5. Show that if $\Phi = \bar{D}\bar{D}U$ for some U, then $\bar{D}_{\dot{\alpha}}\Phi = 0$. Show the converse: for a chiral superfield Φ ($\bar{D}_{\dot{\alpha}}\Phi = 0$), find a U such that $\Phi = \bar{D}\bar{D}U$.

6. A convenient and common way to think about field theory is in terms of a derivative expansion. Let us consider the quantum mechanics of a particle x^i in 9 dimensions so

$i = 1, \ldots 9$. The field $x^i(t)$ transforms as a vector of $SO(9)$. We also introduce 16 real fermions ψ_α, $\alpha = 1, \ldots, 16$ which transform as a spinor of $SO(9)$. Consider the action for a free particle,

$$S = \frac{1}{g^2} \int \left\{ \frac{1}{2} (\dot{x}^i)^2 + \frac{i}{2} \psi_\alpha \dot{\psi}_\alpha \right\}.$$

(i) Check that

$$\delta x^i = -i\epsilon\gamma^i\psi, \qquad \delta\psi_\alpha = (\gamma^i v^i \epsilon)_\alpha$$

define a symmetry of the action. Here γ^i are gamma matrices in nine dimensions satisfying

$$\{\gamma^i, \gamma^j\} = \delta^{ij}.$$

These matrices are symmetric and real (you do not need the explicit form). The Grassmann parameter ϵ is also real. The dimensionful coupling g^2 has mass dimension 3 (it plays no role for the moment but is important in the second part). This is an example of supersymmetry with different numbers of bosons and fermions, and is closely related to the vector superfields we will discuss. Check that the supersymmetry algebra closes. Do you need the equations of motion?

(ii) (Optional open problem - you can work in groups on this question, if you wish, and solutions can be submitted at any time) Even in such a simple system, there are basic open questions which are quite deep. This theory has 16 conserved supersymmetries so there is no (known) convenient superspace. Let us add higher derivative interactions. To do so, we need a quantity that determines a mass scale. In this case, that quantity is the expectation value for the scalars.

$$< (x^i)^2 >= r^2 \neq 0.$$

Let $v^i = \dot{x}^i$. We could imagine adding interactions which vanish as $r \to \infty$. For example,

$$\int \frac{v^4}{r^7} + \ldots.$$

Note this has no powers of the coupling relative to the free action so it is a 1-loop interaction. In a derivative expansion, we lump together all interactions with the same value of

$$n = n_\partial + \frac{1}{2} n_F$$

where n_∂ is the number of derivatives and n_F is the number of fermions in the interaction. So this interaction has $n = 4$. At this same order, there is an interaction

$$\int \frac{v^4}{r^7} + \ldots + f(r)\psi^8.$$

The term $f(r)\psi^8$ means all terms of that form (8 fermions). First show (or look up the proof in hep-th/9805018) that $f(r)$ is actually determined by the free particle supersymmetry transformations.

The interesting issue is the following: once you add $n = 4$ terms, the SUSY algebra is modified. Let us write the modified algebra in the form,

$$\delta x^i = \epsilon \left(\gamma^i + N^i\right) \psi, \qquad \delta\psi_\alpha = (\gamma^i v^i \epsilon + M\epsilon)_\alpha.$$

Here N is $O(2)$ in derivatives while M is $O(3)$ (It is reasonable to assume that N and M contain only positive powers of g^2 to all orders in the derivative expansion.) For appropriate N, M, this algebra closes to $O(4)$. At this order, N and M have been worked out explicitly in hep-th/0210133.

However, the SUSY algebra does not close to higher orders in the derivative expansion. The algebra almost never closes on a finite set of higher derivative interactions. Theories with closed SUSY algebras typically have only terms of $O(2)$ in their EOM. Physically, there must some minimal set of higher derivative terms induced by the ψ^8 interaction and needed for closure of the SUSY algebra. For example a term like ψ^{12}/r^{20} (and its SUSY completion) is required. Can you find a way to describe the minimal set of terms needed for a closed algebra?

For theories with less SUSY, superspace does give a means of obtaining this infinite set of terms since the algebra always closes off-shell. In those cases, the higher derivative interactions make the EOM for the auxiliary field dynamical (as you can check in simple examples), but it can still be solved in a power series in time derivatives and substituted back into the action or supersymmetry transformations.

P.S. There is some progress on this problem in hep-th/0404056 but the complete answer is unknown.