1. We have been remiss! (well, perhaps intentionally). We left out one important representation of a finite group. It is called the regular representation, and it has the same dimension as the order of the group. We define an orthonormal basis for the vector space of this representation by associating a basis vector $|g\rangle$ to each element $g$ of the group $G$. The representation matrix $T(g_1)$ of the group element $g_1$ acts on the basis vector $|g_2\rangle$ in the following way,

$$T(g_1)|g_2\rangle = |g_1g_2\rangle.$$

(i) Construct the regular representation explicitly for our old friend, the dihedral group $D_4$. 

(ii) Is it reducible? If so, how does it reduce into irreps. of $D_4$?

(iii) Lastly, consider the regular representation of a general finite group $G$. Is it reducible? If so, what can you say about its decomposition into irreps. of $G$?

2. We discovered a new particle! Let’s call it the gradstent. It is a very strange particle: it seems to be bosonic but it transforms with spin $1/2$ under an $SU(2)$ symmetry which we will coincidentally call isospin. It also seems to be non-interacting.

(i) Take bosonic annihilation and creation operators $a^\alpha_k, (a^\alpha_k)^\dagger$ where $\alpha$ is a spin $1/2$ label and $k$ labels the gradstent momentum. Fix the momentum to be $k = k_0$ for each annihilation/creation operator. Build the Hilbert space by acting on a Fock vacuum $|0\rangle$. Under which irreps of isospin do the states in your Hilbert space transform? Pick two of those irreps and construct the states transforming in those two particular irreps (be sure you have the right number of states).

(ii) Even more surprisingly, a new paper appeared tonight on the net by a competing group. They claim to have found a gradstentino. A fermionic partner of the gradstent still transforming with spin $1/2$. Perform the same exercise with fermionic annihilation and creation operators: $b^\alpha_{k_0}, (b^\alpha_{k_0})^\dagger$.

(iii) A brilliant experimenter from the EKG collaboration disputes the claim that the gradstent and gradstentino are non-interacting. After long nights in the lab, this experimenter proposes the following Hamiltonian,

$$H = \sum_\alpha E_0 \left[(a^\alpha_{k_0})^\dagger a_{k_0,\alpha} + (b^\alpha_{k_0})^\dagger b_{k_0,\alpha}\right] + g \sum_{\alpha,\beta} (a^\alpha_{k_0})^\dagger a^\beta_{k_0} (b_{k_0,\alpha})^\dagger b_{k_0,\beta},$$

where $g$ is a coupling constant. The $\alpha, \beta$ indices are moved down by using an epsilon tensor $\epsilon_{\alpha\beta}$ with

$$T_\alpha = \epsilon_{\alpha\beta} T^\beta,$$

and where $\epsilon_{12} = \epsilon^{21} = 1$ while $\epsilon_{21} = \epsilon^{21} = -1$. What does this modification do qualitatively? Can you find the eigenstates?
3. To really get a handle on Dynkin diagrams, we need to do a few exercises. Consider the algebra $so(5)$, or in the Dynkin classification, $B_2$.

(i) What is the rank of $so(5)$?

(ii) Construct the Cartan matrix from the Dynkin diagram and draw the simple roots for this algebra.

(iii) Can you find all the positive roots? How many are there? Draw all the roots of $so(5)$.

(iv) How many fundamental representations are there? Can you find the highest weight vectors for these fundamental representations? Please draw them in a diagram where you also include (and label) your simple roots.

(v) Can you find all the remaining weights for the each fundamental representation? What dimension does each representation have?

4. A set of generators for the fundamental representation of $SU(3)$ are given by the Gell-Mann matrices:

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},
\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},
\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{align*}
\]

Actually, it is conventional to take the generators $T_a = \frac{1}{2}\lambda_a$ so $\text{Tr}(T_a T_b) = \frac{1}{2} \delta_{ab}$.

(i) Take $T^3$ and $T^8$ as convenient generators for the Cartan sub-algebra. Find the weights of the fundamental representation. Draw the weights in the $T^3, T^8$ plane.

(ii) Find the roots of $SU(3)$. Again plot the roots in the $T^3, T^8$ plane. Which are the simple roots?

(iii) The first 3 generators $T^1, T^2, T^3$ may look suspiciously familiar. Show that they generator an $SU(2)$ subgroup of $SU(3)$. This subgroup is the isospin subgroup. Every representation of $SU(3)$ is also a (generally reducible) representation of this subgroup. This is true in general. Any irrep can be decomposed into irreps of a subgroup. How does the 3-dimensional representation given by the Gell-Mann matrices transform under this $SU(2)$ subgroup?

(iv) Is there another 3-dimensional representation? If you believe there is, can you find it explicitly and show that it is not equivalent to the one given by the Gell-Mann matrices? Should it exist, what are its weights?

Does the same argument give more than one 2-dimensional representation for $SU(2)$. If not, why not?